(1) (a) Fix \( d \in \mathbb{N} \). Let \( \mathbb{Q}[T]_d \) denote the set of polynomials \( c_0 + c_1 T + \ldots + c_d T^d \) of degree \( \leq d \) with coefficients \( c_0, c_1, \ldots, c_d \) all in \( \mathbb{Q} \). Find a bijection from \( \mathbb{Q}^{d+1} \) to \( \mathbb{Q}[T]_d \). Define \( b(c_0, c_1, \ldots, c_d) = c_0 + c_1 T + \ldots + c_d T^d \).

(b) Fix \( d \in \mathbb{N} \). Prove that \( \mathbb{Q}[T]_d \) is a countable set. Since \( \mathbb{Q} \) is countable, and a finite product of countable sets is countable, \( \mathbb{Q}^{d+1} \) is countable. Since there is a bijection from \( \mathbb{Q}^{d+1} \) to \( \mathbb{Q}[T]_d \), \( \mathbb{Q}[T]_d \) is also countable.

(c) Let \( \mathbb{Q}[T] \) denote the set of all polynomials with coefficients in \( \mathbb{Q} \). Prove that \( \mathbb{Q}[T] \) is countable. \( \mathbb{Q}[T] = \bigcup_{d=0}^{\infty} \mathbb{Q}[T]_d \). Since \( \mathbb{Q}[T]_d \) is countable for all \( d \in \mathbb{N} \), \( \mathbb{Q}[T] \) is a countable union of countable sets, and so is itself countable.
Let \( s \subset \mathbb{R} \) and let \( r \in \mathbb{R}_+ \).
Assume that \( S \) has an upper bound, and that \( S \) has a lower bound, and that \( |x - y| < r \) \( \forall x, y \in S \).
Prove that \( \sup(S) - \inf(S) \leq r \).

Suppose \( \sup(S) - \inf(S) > r \).
Let \( \varepsilon = \frac{1}{2} [\sup(S) - \inf(S) - r] \).
\( \sup(S) - \varepsilon \) is not an upper bound of \( S \), and so
\( \exists x \in S \) such that \( x > \sup(S) - \varepsilon \).
\( \inf(S) + \varepsilon \) is not a lower bound of \( S \), and so
\( \exists y \in S \) such that \( y < \inf(S) + \varepsilon \).
Thus
\( \sup(S) - \varepsilon - (\inf(S) + \varepsilon) < x - y \leq |x - y| < r \).
However
\( \sup(S) - \varepsilon - (\inf(S) + \varepsilon) = \sup(S) - \inf(S) - 2\varepsilon = r \).
Thus we obtain \( r < r \) which is impossible.
(3) Let \( X = \mathbb{R}^2 \), \( Y = \mathbb{R}^2 \),
\[ E_1 = \{(x,y) \in \mathbb{R}^2 : x > 0\} \]
\[ E_2 = \{(x,y) \in \mathbb{R}^2 : x \geq 0\} \]
and \( E_3 = \{(x,y) \in \mathbb{R}^2 : -1 < x < 1 \text{ and } y = 0\} \).

(a) For each \( i \in \{1,2,3\} \), determine whether \( E_i \) is an open subset of \( X \), a closed subset of \( X \), or neither. Prove your answer in each case.

\( E_1 \) is an open subset of \( X \): Let \((x,y) \in E_1\), so \( x > 0 \).
If \((x,y) \in B((x,y); \varepsilon)\), then \( x = x - x + x \)
\[ \geq x - |x - x| \geq x - \sqrt{(x-x)^2 + (y-y)^2} > x - x = 0 \]
\( E_2 \) is a closed subset of \( X \): If \((x_0,y_0)\) is a limit point of \( E_2 \), then \( \forall \varepsilon \in \mathbb{R}^+ \exists (x,y) \in E_2 \) such that \( \sqrt{(x-x_0)^2 + (y-y_0)^2} < \varepsilon \).

Suppose \( x_0 < 0 \). Choose \( \varepsilon = -x_0 \) and choose \((x,y) \in E_2\).

(b) Determine whether \( E_3 \) is an open subset of \( Y \), a closed subset of \( Y \), or neither. Prove your answer.
\( E_3 \) is an open subset of \( Y \): If \((x,y) \in E_3\) then \(-1 < x < 1\) and \( y = 0 \). Let \( r = \min\{|1-x, x+1|\} \). Then \( B_Y((x,y); r) \)
\[ = \{(x,0) \in Y : |x-x| < r\} \subseteq E_3 \]
(3a) continued)  \[ \sqrt{(x-x_0)^2 + (y-y_0)^2} < \varepsilon = -x_0 \]

Then \[ x = x - x_0 + x_0 \leq \sqrt{(x-x_0)^2 + (y-y_0)^2} + x_0 \]
\[ < -x_0 + x_0 = 0, \]
which contradicts \((x, y) \in E_2\). Thus \(x_0 > 0\) and so \((x_0, y_0) \in E_2\).

\(E_3\) is neither an open subset of \(X\) nor a closed subset of \(X\):
\((0,0) \in E_3\) and \(\forall r \in \mathbb{R}^+, B((0,0); r) \not\subseteq E_3\)
because \((0, \frac{1}{2}) \in B((0,0); r)\) but \((0, \frac{1}{2}) \not\in E_3\). This proves that \(E_3\) is not open.

\((1,0) \not\in E_3\) but \(\forall r \in \mathbb{R}^+, E_3 \cap [B((1,0); r) \setminus \{(1,0)\}] \neq \emptyset\)
since if \(-1 < x < 1\) and \(1 - x < r\) then
\((x,0) \in E_3 \cap [B((1,0); r) \setminus \{(1,0)\}]\). This proves that
\(E_3\) is not a closed subset of \(X\).