Surf’s Up: Using Models To Predict Huge Waves

GALVESTON, Feb. 15, 2005—If you are a ship captain and there might be 50-foot waves headed your way, you would appreciate some information about them, right?

That’s the idea behind a wave model system a Texas A&M University at Galveston professor has developed. His detailed wave prediction system is currently in use in the Gulf of Mexico and the Gulf of Maine.

Vijay Panchang, head of the Department of Maritime Systems Engineering, doesn’t make waves; he predicts what they will do, when they will do it, and how high they will get.

Using data provided daily from NOAA and his own complex mathematical models, Panchang and research engineer Doncheng Li provide daily wave model predictions for much of the Texas coast, the Gulf of Mexico, and the Gulf of Maine. Their simulations, updated every 12 hours, provide a forecast for two days ahead.

Because the models use wind data, tsunamis that are created by undersea earthquakes cannot be predicted. But that is not to say his modeling system does not come up with some big waves.

His wave model predicted big waves in November 2003 in the Gulf of Maine, and it was accurate; waves as high as 30 feet were recorded during one storm even in coastal regions.

Last summer during Hurricane Ivan, a buoy located 60 miles south of the Alabama coast recorded a whopping 60-foot wave. “There may have been higher waves because right after recording the 60-foot wave, the buoy snapped and stopped functioning,” he says.


—See the Chapter Project—

A Look Back

In Chapter 3, we began our discussion of functions. We defined domain and range and independent and dependent variables; we found the value of a function and graphed functions. We continued our study of functions by listing properties that a function might have, like being even or odd, and we created a library of functions, naming key functions and listing their properties, including the graph.

A Look Ahead

In this chapter we define the trigonometric functions, six functions that have wide application. We shall talk about their domain and range, see how to find values, graph them, and develop a list of their properties.

There are two widely accepted approaches to the development of the trigonometric functions: one uses right triangles; the other uses circles, especially the unit circle. In this book, we develop the trigonometric functions using right triangles. In Section 7.5, we introduce trigonometric functions using the unit circle and show that this approach leads to the definition using right triangles.
A ray, or half-line, is that portion of a line that starts at a point V on the line and extends indefinitely in one direction. The starting point V of a ray is called its vertex. See Figure 1.

If two rays are drawn with a common vertex, they form an angle. We call one of the rays of an angle the initial side and the other the terminal side. The angle formed is identified by showing the direction and amount of rotation from the initial side to the terminal side. If the rotation is in the counterclockwise direction, the angle is positive; if the rotation is clockwise, the angle is negative. See Figure 2.

Lowercase Greek letters, such as \( \alpha \) (alpha), \( \beta \) (beta), \( \gamma \) (gamma), and \( \theta \) (theta), will often be used to denote angles. Notice in Figure 2(a) that the angle \( \alpha \) is positive because the direction of the rotation from the initial side to the terminal side is counterclockwise. The angle \( \beta \) in Figure 2(b) is negative because the rotation is clockwise. The angle \( \gamma \) in Figure 2(c) is positive. Notice that the angle \( \alpha \) in Figure 2(a) and the angle \( \gamma \) in Figure 2(c) have the same initial side and the same terminal side. However, \( \alpha \) and \( \gamma \) are unequal, because the amount of rotation required to go from the initial side to the terminal side is greater for angle \( \gamma \) than for angle \( \alpha \).

An angle \( \theta \) is said to be in standard position if its vertex is at the origin of a rectangular coordinate system and its initial side coincides with the positive x-axis. See Figure 3.
SECTION 7.1 Angles and Their Measure

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When an angle $\theta$ is in standard position, the terminal side will lie either in a quadrant, in which case we say that $\theta$ lies in that quadrant, or will lie on the $x$-axis or the $y$-axis, in which case we say that $\theta$ is a quadrantal angle. For example, the angle $\theta$ in Figure 4(a) lies in quadrant II, the angle $\theta$ in Figure 4(b) lies in quadrant IV, and the angle $\theta$ in Figure 4(c) is a quadrantal angle.

We measure angles by determining the amount of rotation needed for the initial side to become coincident with the terminal side. The two commonly used measures for angles are degrees and radians.

**Degrees**

The angle formed by rotating the initial side exactly once in the counterclockwise direction until it coincides with itself (1 revolution) is said to measure 360 degrees, abbreviated 360°. One degree, 1°, is $\frac{1}{360}$ revolution. A right angle is an angle that measures 90°, or $\frac{1}{4}$ revolution; a straight angle is an angle that measures 180°, or $\frac{1}{2}$ revolution. See Figure 5. As Figure 5(b) shows, it is customary to indicate a right angle by using the symbol $\Box$.

It is also customary to refer to an angle that measures $\theta$ degrees as an angle of $\theta$ degrees.

**EXAMPLE 1**

**Drawing an Angle**

Draw each angle.

(a) 45°  (b) −90°  (c) 225°  (d) 405°

**Solution**

(a) An angle of 45° is $\frac{1}{2}$ of a right angle. See Figure 6.

(b) An angle of −90° is $\frac{1}{4}$ revolution in the clockwise direction. See Figure 7.
(c) An angle of 225° consists of a rotation through 180° followed by a rotation through 45°. See Figure 8.

(d) An angle of 405° consists of 1 revolution (360°) followed by a rotation through 45°. See Figure 9.

**Figure 8**

**Figure 9**

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**Problem 11**

**1 Convert between Decimals and Degrees, Minutes, Seconds Forms for Angles**

Although subdivisions of a degree may be obtained by using decimals, we also may use the notion of minutes and seconds. **One minute**, denoted by 1’, is defined as \( \frac{1}{60} \) degree.

**One second**, denoted by 1”, is defined as \( \frac{1}{60} \) minute, or equivalently, \( \frac{1}{3600} \) degree.

An angle of, say, 30 degrees, 40 minutes, 10 seconds is written compactly as 30°40’10”.

To summarize:

\[
1 \text{ counterclockwise revolution} = 360° \\
1° = 60' \\
1' = 60''
\]

(1)

It is sometimes necessary to convert from the degree, minute, second notation (D°M’S”) to a decimal form, and vice versa. Check your calculator; it should be capable of doing the conversion for you.

Before getting started, though, you must set the mode to degrees because there are two common ways to measure angles: degree mode and radian mode. (We will define radians shortly.) Usually, a menu is used to change from one mode to another. Check your owner’s manual to find out how your particular calculator works.

Now let’s see how to convert from the degree, minute, second notation (D°M’S”) to a decimal form, and vice versa, by looking at some examples:

\[
15°30’ = 15.5° \quad \text{because} \quad 30’ = 30 \cdot 1’ = 30 \cdot \left( \frac{1}{60} \right)^{°} = 0.5°
\]

\[
1’ = \left( \frac{1}{60} \right)^{°}
\]

\[
32.25° = 32°15’ \quad \text{because} \quad 0.25° = \left( \frac{1}{4} \right)^{°} = \frac{1}{4} \cdot 1° = \frac{1}{4} (60’) = 15’
\]

\[
1° = 60’
\]

**Example 2**

**Converting between Degrees, Minutes, Seconds Form and Decimal Form**

(a) Convert 50°6’21” to a decimal in degrees. Round the answer to four decimal places.

(b) Convert 21.256° to the D°M’S” form. Round the answer to the nearest second.
Solution

(a) Because 1\(^{\circ}\) = \left(\frac{1}{60}\right)^{\circ} and 1\(^{\circ}\) = \left(\frac{1}{60}\right)^{\prime} = \left(\frac{1}{60} \cdot \frac{1}{60}\right)^{\circ}, we convert as follows:

\[
50^\circ 6' 21'' = 50^\circ + 6^\prime + 21''
= 50^\circ + 6 \cdot \left(\frac{1}{60}\right)^{\circ} + 21 \cdot \left(\frac{1}{60} \cdot \frac{1}{60}\right)^{\circ}
\approx 50^\circ + 0.1^\circ + 0.0058^\circ
= 50.1058^\circ
\]

(b) We proceed as follows:

\[
21.256^\circ = 21^\circ + 0.256^\circ
= 21^\circ + (0.256)(60')
= 21^\circ + 15.36'
= 21^\circ + 15' + 0.36'
= 21^\circ + 15' + (0.36)(60'')
= 21^\circ + 15' + 21.6''
\approx 21^\circ 15' 22''
\]

In many applications, such as describing the exact location of a star or the precise position of a ship at sea, angles measured in degrees, minutes, and even seconds are used. For calculation purposes, these are transformed to decimal form. In other applications, especially those in calculus, angles are measured using radians.

Radian

A central angle is a positive angle whose vertex is at the center of a circle. The rays of a central angle subtend (intersect) an arc on the circle. If the radius of the circle is \(r\) and the length of the arc subtended by the central angle is also \(r\), then the measure of the angle is 1 radian. See Figure 10(a).

For a circle of radius 1, the rays of a central angle with measure 1 radian would subtend an arc of length 1. For a circle of radius 3, the rays of a central angle with measure 1 radian would subtend an arc of length 3. See Figure 10(b).

2 Find the Arc Length of a Circle

Now consider a circle of radius \(r\) and two central angles, \(\theta\) and \(\theta_1\), measured in radians. Suppose that these central angles subtend arcs of lengths \(s\) and \(s_1\), respectively,
as shown in Figure 11. From geometry, we know that the ratio of the measures of the angles equals the ratio of the corresponding lengths of the arcs subtended by these angles; that is,

$$\frac{\theta}{\theta_1} = \frac{s}{s_1} \quad (2)$$

Suppose that $\theta_1 = 1$ radian. Refer again to Figure 10(a). The length $s_1$ of the arc subtended by the central angle $\theta_1 = 1$ radian equals the radius $r$ of the circle. Then $s_1 = r$, so equation (2) reduces to

$$\frac{\theta}{1} = \frac{s}{r} \quad \text{or} \quad s = r\theta \quad (3)$$

**THEOREM**

**Arc Length**

For a circle of radius $r$, a central angle of $\theta$ radians subtends an arc whose length $s$ is

$$s = r\theta \quad (4)$$

**NOTE** Formulas must be consistent with regard to the units used. In equation (4), we write $s = r\theta$.

To see the units, however, we must go back to equation (3) and write

$$\frac{\theta \text{ radians}}{1 \text{ radian}} = \frac{s \text{ length units}}{r \text{ length units}}$$

$$s \text{ length units} = r \text{ length units} \cdot \frac{\theta \text{ radians}}{1 \text{ radian}}$$

Since the radians cancel, we are left with

$$s \text{ length units} = (r \text{ length units})\theta \quad s = r\theta$$

where $\theta$ appears to be “dimensionless” but, in fact, is measured in radians. So, in using the formula $s = r\theta$, the dimension for $\theta$ is radians, and any convenient unit of length (such as inches or meters) may be used for $s$ and $r$.

**EXAMPLE 3** Finding the Length of an Arc of a Circle

Find the length of the arc of a circle of radius 2 meters subtended by a central angle of 0.25 radian.

**Solution**

We use equation (4) with $r = 2$ meters and $\theta = 0.25$. The length $s$ of the arc is

$$s = r\theta = 2(0.25) = 0.5 \text{ meter}$$

Now Work **PROBLEM 71**

3 Convert from Degrees to Radians and from Radians to Degrees

Next we discuss the relationship between angles measured in degrees and angles measured in radians. Consider a circle of radius $r$. A central angle of 1 revolution will subtend an arc equal to the circumference of the circle (Figure 12). Because the circumference of a circle equals $2\pi r$, we use $s = 2\pi r$ in equation (4) to find that, for an angle $\theta$ of 1 revolution,

$$s = r\theta$$

$$2\pi r = r\theta$$

$$\theta = 2\pi \text{ radians}$$

Solve for $\theta$. 

![Figure 11](image1.png)

![Figure 12](image2.png)
From this we have,

\[
1 \text{ revolution} = 2\pi \text{ radians} \quad (5)
\]

Since 1 revolution = 360°, we have

\[
360° = 2\pi \text{ radians}
\]

Dividing both sides by 2 yields

\[
180° = \pi \text{ radians} \quad (6)
\]

Divide both sides of equation (6) by 180. Then

\[
1 \text{ degree} = \frac{\pi}{180} \text{ radian}
\]

Divide both sides of (6) by \(\pi\). Then

\[
\frac{180}{\pi} \text{ degrees} = 1 \text{ radian}
\]

We have the following two conversion formulas:

\[
1 \text{ degree} = \frac{\pi}{180} \text{ radian} \quad 1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \quad (7)
\]

**EXAMPLE 4**

**Converting from Degrees to Radians**

Convert each angle in degrees to radians.

(a) 60°  (b) 150°  (c) –45°  (d) 90°  (e) 107°

**Solution**

(a) 60° = 60 \cdot 1 \text{ degree} = 60 \cdot \frac{\pi}{180} \text{ radian} = \frac{\pi}{3} \text{ radians}

(b) 150° = 150 \cdot 1° = 150 \cdot \frac{\pi}{180} \text{ radian} = \frac{5\pi}{6} \text{ radians}

(c) –45° = –45 \cdot \frac{\pi}{180} \text{ radian} = –\frac{\pi}{4} \text{ radian}

(d) 90° = 90 \cdot \frac{\pi}{180} \text{ radian} = \frac{\pi}{2} \text{ radians}

(e) 107° = 107 \cdot \frac{\pi}{180} \text{ radian} \approx 1.868 \text{ radians}

Example 4, parts (a)–(d), illustrates that angles that are “nice” fractions of a revolution are expressed in radian measure as fractional multiples of \(\pi\), rather than as decimals. For example, a right angle, as in Example 4(d), is left in the form \(\frac{\pi}{2}\) radians, which is exact, rather than using the approximation \(\frac{\pi}{2} \approx \frac{3.1416}{2} = 1.5708\) radians. When the fractions are not “nice,” we use the decimal approximation of the angle, as in Example 4(e).

**New Work Problems 35 and 61**
**EXAMPLE 5**

Converting Radians to Degrees

Convert each angle in radians to degrees.

(a) \( \frac{\pi}{6} \) radian
(b) \( \frac{3\pi}{2} \) radians
(c) \(-\frac{3\pi}{4} \) radians
(d) \( \frac{7\pi}{3} \) radians
(e) 3 radians

**Solution**

(a) \( \frac{\pi}{6} \) radian = \( \frac{\pi}{6} \cdot \frac{180}{\pi} \) degrees = \( 30^\circ \)
(b) \( \frac{3\pi}{2} \) radians = \( \frac{3\pi}{2} \cdot \frac{180}{\pi} \) degrees = \( 270^\circ \)
(c) \(-\frac{3\pi}{4} \) radians = \( -\frac{3\pi}{4} \cdot \frac{180}{\pi} \) degrees = \(-135^\circ \)
(d) \( \frac{7\pi}{3} \) radians = \( \frac{7\pi}{3} \cdot \frac{180}{\pi} \) degrees = \( 420^\circ \)
(e) 3 radians = \( 3 \cdot \frac{180}{\pi} \) degrees \( \approx 171.89^\circ \)

**Table 1**

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<th>Degrees</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>135°</th>
<th>150°</th>
<th>180°</th>
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<td>( \frac{3\pi}{4} )</td>
<td>( \frac{5\pi}{6} )</td>
<td>( \pi )</td>
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<td>225°</td>
<td>240°</td>
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<td>300°</td>
<td>315°</td>
<td>330°</td>
<td>360°</td>
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<tr>
<td>Radians</td>
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<td>( \frac{5\pi}{4} )</td>
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<td>( \frac{11\pi}{6} )</td>
<td>( 2\pi )</td>
<td></td>
</tr>
</tbody>
</table>

**EXAMPLE 6**

Finding the Distance between Two Cities

See Figure 13(a). The latitude of a location \( L \) is the angle formed by a ray drawn from the center of Earth to the Equator and a ray drawn from the center of Earth to \( L \). See Figure 13(b). Glasgow, Montana, is due north of Albuquerque, New Mexico. Find the distance between Glasgow (48°9' north latitude) and Albuquerque (35°5' north latitude). Assume that the radius of Earth is 3960 miles.
The measure of the central angle between the two cities is $48^\circ 9' - 35^\circ 5' = 13^\circ 4'$.

We use equation (4), $s = r\theta$, but first we must convert the angle of $13^\circ 4'$ to radians.

$$\theta = 13^\circ 4' \approx 13.0667^\circ = 13.0667 \cdot \frac{\pi}{180} \text{ radian} \approx 0.228 \text{ radian}$$

We use $\theta = 0.228$ radian and $r = 3960$ miles in equation (4). The distance between the two cities is

$$s = r\theta = 3960 \cdot 0.228 \approx 903 \text{ miles}$$

When an angle is measured in degrees, the degree symbol will always be shown. However, when an angle is measured in radians, we will follow the usual practice and omit the word \textit{radians}. So, if the measure of an angle is given as it is understood to mean \textit{radian}.

### Finding the Area of a Sector of a Circle

Consider a circle of radius $r$. Suppose that $\theta$, measured in radians, is a central angle of this circle. See Figure 14. We seek a formula for the area $A$ of the sector (shown in blue) formed by the angle $\theta$.

Now consider a circle of radius $r$ and two central angles $\theta$ and $\theta_1$, both measured in radians. See Figure 15. From geometry, we know the ratio of the measures of the angles equals the ratio of the corresponding areas of the sectors formed by these angles. That is,

$$\frac{\theta}{\theta_1} = \frac{A}{A_1}$$

Suppose that $\theta_1 = 2\pi$ radians. Then $A_1 = \text{area of the circle} = \pi r^2$. Solving for $A$, we find

$$A = A_1 \frac{\theta}{\theta_1} = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$$

The area $A$ of the sector of a circle of radius $r$ formed by a central angle of $\theta$ radians is

$$A = \frac{1}{2} r^2 \theta$$

### Example 7

**Finding the Area of a Sector of a Circle**

Find the area of the sector of a circle of radius 2 feet formed by an angle of $30^\circ$. Round the answer to two decimal places.

**Solution**

We use equation (8) with $r = 2$ feet and $\theta = 30^\circ = \frac{\pi}{6}$ radian. [Remember, in equation (8), $\theta$ must be in radians.]

$$A = \frac{1}{2} r^2 \theta = \frac{1}{2} (2)^2 \frac{\pi}{6} = \frac{\pi}{3}$$

The area $A$ of the sector is 1.05 square feet, rounded to two decimal places.
5 Find the Linear Speed of an Object Traveling in Circular Motion

We have already defined the average speed of an object as the distance traveled divided by the elapsed time.

Suppose that an object moves around a circle of radius \( r \) at a constant speed. If \( s \) is the distance traveled in time \( t \) around this circle, then the linear speed \( v \) of the object is defined as

\[
v = \frac{s}{t} \tag{9}
\]

As this object travels around the circle, suppose that \( \theta \) (measured in radians) is the central angle swept out in time \( t \). See Figure 16.

The angular speed \( \omega \) (the Greek letter omega) of this object is the angle \( \theta \) (measured in radians) swept out, divided by the elapsed time \( t \), that is,

\[
\omega = \frac{\theta}{t} \tag{10}
\]

Angular speed is the way the turning rate of an engine is described. For example, an engine idling at 900 rpm (revolutions per minute) is one that rotates at an angular speed of

\[
900 \text{ revolutions/minute} = 900 \text{ revolutions/minute} \cdot \frac{2\pi \text{ radians}}{\text{revolution}} = 1800\pi \text{ radians/minute}
\]

There is an important relationship between linear speed and angular speed:

\[
\text{linear speed} = v = \frac{s}{t} = \frac{r\theta}{t} = r \left( \frac{\theta}{t} \right) = r \cdot \omega
\]

So,

\[
v = r\omega \tag{11}
\]

where \( \omega \) is measured in radians per unit time.

When using equation (11), remember that \( v = \frac{s}{t} \) (the linear speed) has the dimensions of length per unit of time (such as feet per second or miles per hour), \( r \) (the radius of the circular motion) has the same length dimension as \( s \), and \( \omega \) (the angular speed) has the dimensions of radians per unit of time. If the angular speed is given in terms of revolutions per unit of time (as is often the case), be sure to convert it to radians per unit of time before attempting to use equation (11). Remember, 1 revolution = \( 2\pi \) radians.

**Example 8** Finding Linear Speed

A child is spinning a rock at the end of a 2-foot rope at the rate of 180 revolutions per minute (rpm). Find the linear speed of the rock when it is released.

**Solution**

Look at Figure 17. The rock is moving around a circle of radius \( r = 2 \) feet. The angular speed \( \omega \) of the rock is

\[
\omega = 180 \text{ revolutions/minute} = 180 \text{ revolutions/minute} \cdot \frac{2\pi \text{ radians}}{\text{revolution}} = 360\pi \text{ radians/minute}
\]
From equation (11), the linear speed $v$ of the rock is

$$v = r\omega = 2 \text{ feet} \cdot 360\pi \text{ radians} = 720\pi \text{ feet per minute} \approx 2262 \text{ feet per minute}$$

The linear speed of the rock when it is released is $2262 \text{ ft/min} \approx 25.7 \text{ mi/hr.}$

**Now Work Problem 97**

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### Historical Feature

Trigonometry was developed by Greek astronomers, who regarded the sky as the inside of a sphere, so it was natural that triangles on a sphere were investigated early (by Menelaus of Alexandria about AD 100) and that triangles in the plane were studied much later. The first book containing a systematic treatment of plane and spherical trigonometry was written by the Persian astronomer Nasir Eddin (about AD 1250). Regiomontanus (1436–1476) is the person most responsible for moving trigonometry from astronomy into mathematics. His work was improved by Copernicus (1473–1543) and Copernicus’s student Rhaeticus (1514–1576). Rhaeticus’s book was the first to define the six trigonometric functions as ratios of sides of triangles, although he did not give the functions their present names. Credit for this is due to Thomas Finck (1583), but Finck’s notation was by no means universally accepted at the time. The notation was finally stabilized by the textbooks of Leonhard Euler (1707–1783). Trigonometry has since evolved from its use by surveyors, navigators, and engineers to present applications involving ocean tides, the rise and fall of food supplies in certain ecologies, brain wave patterns, and many other phenomena.

### 7.1 Assess Your Understanding

**‘Are You Prepared?’ Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.**

1. What is the formula for the circumference $C$ of a circle of radius $r$? (pp. 31–32)
2. What is the formula for the area $A$ of a circle of radius $r$? (pp. 31–32)

**Concepts and Vocabulary**

3. An angle $\theta$ is in _______ if its vertex is at the origin of a rectangular coordinate system and its initial side coincides with the positive $x$-axis.
4. On a circle of radius $r$, a central angle of $\theta$ radians subtends an arc of length $s = \____$; the area of the sector formed by this angle $\theta$ is $A = \____$.
5. An object travels around a circle of radius $r$ with constant speed. If $s$ is the distance traveled in time $t$ around the circle and $\theta$ is the central angle (in radians) swept out in time $t$, then the linear speed of the object is $v = \____$ and the angular speed of the object is $\omega = \____$.

**Skill Building**

In Problems 11–22, draw each angle.

11. $30^\circ$  
12. $60^\circ$  
13. $135^\circ$  
14. $-120^\circ$  
15. $450^\circ$  
16. $540^\circ$

17. $\frac{3\pi}{4}$  
18. $\frac{4\pi}{3}$  
19. $-\frac{\pi}{6}$  
20. $-\frac{2\pi}{3}$  
21. $\frac{16\pi}{3}$  
22. $\frac{21\pi}{4}$

In Problems 23–28, convert each angle to a decimal in degrees. Round your answer to two decimal places.

23. $40^\circ10'25''$  
24. $61^\circ42'21''$  
25. $1^\circ2'3''$  
26. $73^\circ40'40''$  
27. $9^\circ9'9''$  
28. $98^\circ22'45''$

In Problems 29–34, convert each angle to $D'M'S''$ form. Round your answer to the nearest second.

29. $40.32^\circ$  
30. $61.24^\circ$  
31. $18.255^\circ$  
32. $29.411^\circ$  
33. $19.99^\circ$  
34. $44.01^\circ$
CHAPTER 7  Trigonometric Functions

In Problems 35–46, convert each angle in degrees to radians. Express your answer as a multiple of π.

35. 30°  36. 120°  37. 240°  38. 330°  39. −60°  40. −30°

41. 180°  42. 270°  43. −135°  44. −225°  45. −90°  46. −180°

In Problems 47–58, convert each angle in radians to degrees.

47. \(\frac{\pi}{3}\)  48. \(\frac{5\pi}{6}\)  49. \(-\frac{5\pi}{4}\)  50. \(-\frac{2\pi}{3}\)  51. \(\frac{\pi}{2}\)  52. \(4\pi\)

53. \(\frac{\pi}{12}\)  54. \(\frac{5\pi}{12}\)  55. \(-\frac{\pi}{2}\)  56. \(-\pi\)  57. \(-\frac{\pi}{6}\)  58. \(-\frac{3\pi}{4}\)

In Problems 59–64, convert each angle in degrees to radians. Express your answer in decimal form, rounded to two decimal places.

59. 17°  60. 73°  61. −40°  62. −51°  63. 125°  64. 350°

In Problems 65–70, convert each angle in radians to degrees. Express your answer in decimal form, rounded to two decimal places.

65. 3.14  66. 0.75  67. 2  68. 3  69. 6.32  70. \(\sqrt{2}\)

In Problems 71–78, \(s\) denotes the length of the arc of a circle of radius \(r\) subtended by the central angle \(\theta\). Find the missing quantity. Round answers to three decimal places.

71. \(r = 10\) meters, \(\theta = \frac{1}{2}\) radian, \(s = ?\)

72. \(r = 6\) feet, \(\theta = 2\) radians, \(s = ?\)

73. \(\theta = \frac{1}{3}\) radian, \(s = 2\) feet, \(r = ?\)

74. \(\theta = \frac{1}{4}\) radian, \(s = 6\) centimeters, \(r = ?\)

75. \(r = 5\) miles, \(s = 3\) miles, \(\theta = ?\)

76. \(r = 6\) meters, \(s = 8\) meters, \(\theta = ?\)

77. \(r = 2\) inches, \(\theta = 30°\), \(s = ?\)

78. \(r = 3\) meters, \(\theta = 120°\), \(s = ?\)

In Problems 79–86, \(A\) denotes the area of the sector of a circle of radius \(r\) formed by the central angle \(\theta\). Find the missing quantity. Round answers to three decimal places.

79. \(r = 10\) meters, \(\theta = \frac{1}{2}\) radian, \(A = ?\)

80. \(r = 6\) feet, \(\theta = 2\) radians, \(A = ?\)

81. \(\theta = \frac{1}{3}\) radian, \(A = 2\) square feet, \(r = ?\)

82. \(\theta = \frac{1}{4}\) radian, \(A = 6\) square centimeters, \(r = ?\)

83. \(r = 5\) miles, \(A = 3\) square miles, \(\theta = ?\)

84. \(r = 6\) meters, \(A = 8\) square meters, \(\theta = ?\)

85. \(r = 2\) inches, \(\theta = 30°\), \(A = ?\)

86. \(r = 3\) meters, \(\theta = 120°\), \(A = ?\)

In Problems 87–90, find the length \(s\) and area \(A\). Round answers to three decimal places.

87. 88. 89. 90.

Applications and Extensions

91. Movement of a Minute Hand  The minute hand of a clock is 6 inches long. How far does the tip of the minute hand move in 15 minutes? How far does it move in 25 minutes? Round answers to two decimal places.

92. Movement of a Pendulum  A pendulum swings through an angle of 20° each second. If the pendulum is 40 inches long, how far does its tip move each second? Round answers to two decimal places.

93. Area of a Sector  Find the area of the sector of a circle of radius 4 meters formed by an angle of 45°. Round the answer to two decimal places.

94. Area of a Sector  Find the area of the sector of a circle of radius 3 centimeters formed by an angle of 60°. Round the answer to two decimal places.
95. Watering a Lawn  A water sprinkler sprays water over a distance of 30 feet while rotating through an angle of 135°. What area of lawn receives water?

96. Designing a Water Sprinkler  An engineer is asked to design a water sprinkler that will cover a field of 100 square yards that is in the shape of a sector of a circle of radius 50 yards. Through what angle should the sprinkler rotate?

97. Motion on a Circle  An object is traveling around a circle with a radius of 5 centimeters. If in 20 seconds a central angle of \( \frac{1}{3} \) radian is swept out, what is the angular speed of the object? What is its linear speed?

98. Motion on a Circle  An object is traveling around a circle with a radius of 2 meters. If in 20 seconds the object travels 5 meters, what is its angular speed? What is its linear speed?

99. Bicycle Wheels  The diameter of each wheel of a bicycle is 26 inches. If you are traveling at a speed of 35 miles per hour on this bicycle, through how many revolutions per minute are the wheels turning?

100. Car Wheels  The radius of each wheel of a car is 15 inches. If the wheels are turning at the rate of 3 revolutions per second, how fast is the car moving? Express your answer in inches per second and in miles per hour.

In Problems 101–104, the latitude of a location L is the angle formed by a ray drawn from the center of Earth to the Equator and a ray drawn from the center of Earth to L. See the figure.

101. Distance between Cities  Memphis, Tennessee, is due north of New Orleans, Louisiana. Find the distance between Memphis (35°9’ north latitude) and New Orleans (29°57’ north latitude). Assume that the radius of Earth is 3960 miles.

102. Distance between Cities  Charleston, West Virginia, is due north of Jacksonville, Florida. Find the distance between Charleston (38°21’ north latitude) and Jacksonville (30°20’ north latitude). Assume that the radius of Earth is 3960 miles.

103. Linear Speed on Earth  Earth rotates on an axis through its poles. The distance from the axis to a location on Earth 30° north latitude is about 3429.5 miles. Therefore, a location on Earth at 30° north latitude is spinning on a circle of radius 3429.5 miles. Compute the linear speed on the surface of Earth at 30° north latitude.

104. Linear Speed on Earth  Earth rotates on an axis through its poles. The distance from the axis to a location on Earth 40° north latitude is about 3033.5 miles. Therefore, a location on Earth at 40° north latitude is spinning on a circle of radius 3033.5 miles. Compute the linear speed on the surface of Earth at 40° north latitude.

105. Speed of the Moon  The mean distance of the Moon from Earth is Assuming that the orbit of the Moon around Earth is circular and that 1 revolution takes 27.3 days, find the linear speed of the Moon. Express your answer in miles per hour.

106. Speed of Earth  The mean distance of Earth from the Sun is Assuming that the orbit of Earth around the Sun is circular and that 1 revolution takes 365 days, find the linear speed of Earth. Express your answer in miles per hour.

107. Pulleys  Two pulleys, one with radius 2 inches and the other with radius 8 inches, are connected by a belt. (See the figure.) If the 2-inch pulley is caused to rotate at 3 revolutions per minute, determine the revolutions per minute of the 8-inch pulley.

[Hint: The linear speeds of the pulleys are the same; both equal the speed of the belt.]

108. Ferris Wheels  A neighborhood carnival has a Ferris wheel whose radius is 30 feet. You measure the time it takes for one revolution to be 70 seconds. What is the linear speed (in feet per second) of this Ferris wheel? What is the angular speed in radians per second?

109. Computing the Speed of a River Current  To approximate the speed of the current of a river, a circular paddle wheel with radius 4 feet is lowered into the water. If the current causes the wheel to rotate at a speed of 10 revolutions
per minute, what is the speed of the current? Express your answer in miles per hour.

10. **Spin Balancing Tires** A spin balancer rotates the wheel of a car at 480 revolutions per minute. If the diameter of the wheel is 26 inches, what road speed is being tested? Express your answer in miles per hour. At how many revolutions per minute should the balancer be set to test a road speed of 80 miles per hour?

11. **The Cable Cars of San Francisco** At the Cable Car Museum you can see the four cable lines that are used to pull cable cars up and down the hills of San Francisco. Each cable travels at a speed of 9.55 miles per hour, caused by a rotating wheel whose diameter is 8.5 feet. How fast is the wheel rotating? Express your answer in revolutions per minute.

12. **Difference in Time of Sunrise** Naples, Florida, is approximately 90 miles due west of Ft. Lauderdale. How much sooner would a person in Ft. Lauderdale first see the rising Sun than a person in Naples? See the hint.

13. **Keeping Up with the Sun** How fast would you have to travel on the surface of Earth at the equator to keep up with the Sun (that is, so that the Sun would appear to remain in the same position in the sky)?

114. **Nautical Miles** A nautical mile equals the length of arc subtended by a central angle of 1 minute on a great circle* on the surface of Earth. (See the figure.) If the radius of Earth is taken as 3960 miles, express 1 nautical mile in terms of ordinary, or statute, miles.

115. **Approximating the Circumference of Earth** Eratosthenes of Cyrene (276–194 bc) was a Greek scholar who lived and worked in Cyrene and Alexandria. One day while visiting in Syene he noticed that the Sun’s rays shone directly down a well. On this date 1 year later, in Alexandria, which is 500 miles due north of Syene he measured the angle of the Sun to be about 7.2 degrees. See the figure. Use this information to approximate the radius and circumference of Earth.

116. **Designing a Little League Field** For a 60-foot Little League Baseball field, the distance from home base to the nearest fence (or other obstruction) on fair territory should be a minimum of 200 feet. The commissioner of parks and recreation is making plans for a new 60-foot field. Because of limited ground availability, he will use the minimum required distance to the outfield fence. To increase safety, however, he plans to include a 10-foot wide warning track on the inside of the fence. To further increase safety, the fence and warning track will extend both directions into foul territory. In total the arc formed by the outfield fence (including the extensions into the foul territories) will be subtended by a central angle at home plate measuring 96°, as illustrated.

(a) Determine the length of the outfield fence.
(b) Determine the area of the warning track.

* Any circle drawn on the surface of Earth that divides Earth into two equal hemispheres.
SECTION 7.2 Right Triangle Trigonometry

PREPARING FOR THIS SECTION
Before getting started, review the following:

• Geometry Essentials (Chapter R, Review, Section R.3, pp. 30–35)
• Functions (Section 3.1, pp. 208–218)

Now Work the 'Are You Prepared?' problems on page 525.

OBJECTIVES

1. Find the Values of Trigonometric Functions of Acute Angles (p. 517)
2. Use the Fundamental Identities (p. 519)
3. Find the Values of the Remaining Trigonometric Functions, Given the Value of One of Them (p. 521)
4. Use the Complementary Angle Theorem (p. 523)

1. Find the Values of Trigonometric Functions of Acute Angles

A triangle in which one angle is a right angle (90°) is called a right triangle. Recall that the side opposite the right angle is called the hypotenuse, and the remaining two sides are called the legs of the triangle. In Figure 18 we have labeled the hypotenuse as c to indicate that its length is c units, and, in a like manner, we have labeled the legs as a and b. Because the triangle is a right triangle, the Pythagorean Theorem tells us that

\[c^2 = a^2 + b^2\]

Note: There is a 90° angle between the two foul lines. Then there are two 3° angles between the foul lines and the dotted lines shown. The angle between the two dotted lines outside the 200 foot foul lines is 96°.

117. Pulleys

Two pulleys, one with radius \(r_1\) and the other with radius \(r_2\), are connected by a belt. The pulley with radius \(r_1\) rotates at \(\omega_1\) revolutions per minute, whereas the pulley with radius \(r_2\) rotates at \(\omega_2\) revolutions per minute. Show that

\[
\frac{r_1}{r_2} = \frac{\omega_2}{\omega_1}
\]

118. Do you prefer to measure angles using degrees or radians? Provide justification and a rationale for your choice.

119. Which angle has the larger measure: 1 degree or 1 radian? Or are they equal?

120. Explain the difference between linear speed and angular speed.

121. For a circle of radius \(r\), a central angle of \(\theta\) degrees subtends an arc whose length \(s\) is \(s = \frac{\pi}{180} r \theta\). Discuss whether this is a true or false statement. Give reasons to defend your position.

122. Discuss why ships and airplanes use nautical miles to measure distance. Explain the difference between a nautical mile and a statute mile.

123. Investigate the way that speed bicycles work. In particular, explain the differences and similarities between 5-speed and 9-speed derailleurs. Be sure to include a discussion of linear speed and angular speed.

124. In Example 6, we found that the distance between Albuquerque, New Mexico, and Glasgow, Montana, is approximately 903 miles. According to mapquest.com, the distance is approximately 1300 miles. What might account for the difference?
Now, suppose that \( \theta \) is an \textbf{acute angle}; that is, \( 0^\circ < \theta < 90^\circ \) (if \( \theta \) is measured in degrees) and \( 0 < \theta < \frac{\pi}{2} \) (if \( \theta \) is measured in radians). See Figure 19(a). Using this acute angle \( \theta \), we can form a right triangle, like the one illustrated in Figure 19(b), with hypotenuse of length \( c \) and legs of lengths \( a \) and \( b \). Using the three sides of this triangle, we can form exactly six ratios:

\[
\begin{align*}
\frac{b}{c}, & \quad \frac{a}{c}, & \quad \frac{b}{a}, & \quad \frac{c}{b}, & \quad \frac{c}{a}, & \quad \frac{a}{b}
\end{align*}
\]

Figure 19

In fact, these ratios depend only on the size of the angle \( \theta \) and not on the triangle formed. To see why, look at Figure 19(c). Any two right triangles formed using the angle \( \theta \) will be similar and, hence, corresponding ratios will be equal. As a result,

\[
\frac{b}{c} = \frac{b'}{c'}, \quad \frac{a}{c} = \frac{a'}{c'}, \quad \frac{b}{a} = \frac{b'}{a'} = \frac{b'}{a'}, \quad \frac{c}{b} = \frac{c'}{b'} = \frac{c'}{b'}, \quad \frac{c}{a} = \frac{c'}{a'} = \frac{c'}{a}, \quad \frac{a}{b} = \frac{a'}{b'}
\]

Because the ratios depend only on the angle \( \theta \) and not on the triangle itself, we give each ratio a name that involves \( \theta \): sine of \( \theta \), cosine of \( \theta \), tangent of \( \theta \), cosecant of \( \theta \), secant of \( \theta \), and cotangent of \( \theta \).

The six ratios of a right triangle are called \textbf{trigonometric functions of acute angles} and are defined as follows:

<table>
<thead>
<tr>
<th>Function Name</th>
<th>Abbreviation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>sine of ( \theta )</td>
<td>( \sin \theta )</td>
<td>( \frac{b}{c} )</td>
</tr>
<tr>
<td>cosine of ( \theta )</td>
<td>( \cos \theta )</td>
<td>( \frac{a}{c} )</td>
</tr>
<tr>
<td>tangent of ( \theta )</td>
<td>( \tan \theta )</td>
<td>( \frac{b}{a} )</td>
</tr>
<tr>
<td>cosecant of ( \theta )</td>
<td>( \csc \theta )</td>
<td>( \frac{c}{b} )</td>
</tr>
<tr>
<td>secant of ( \theta )</td>
<td>( \sec \theta )</td>
<td>( \frac{c}{a} )</td>
</tr>
<tr>
<td>cotangent of ( \theta )</td>
<td>( \cot \theta )</td>
<td>( \frac{a}{b} )</td>
</tr>
</tbody>
</table>

Figure 20

As an aid to remembering these definitions, it may be helpful to refer to the lengths of the sides of the triangle by the names \textit{hypotenuse} \( c \), \textit{opposite} \( b \), and \textit{adjacent} \( a \). See Figure 20. In terms of these names, we have the following ratios:

\[
\begin{align*}
\sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}, & \cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}, & \tan \theta &= \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a} \\
\csc \theta &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b}, & \sec \theta &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a}, & \cot \theta &= \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}
\end{align*}
\]

(1)

Since \( a, b, \) and \( c \) are positive, each of the trigonometric functions of an acute angle \( \theta \) is positive.
Finding the Value of Trigonometric Functions

Find the value of each of the six trigonometric functions of the angle $\theta$ in Figure 21.

We see in Figure 21 that the two given sides of the triangle are

$$c = \text{hypotenuse} = 5 \quad a = \text{adjacent} = 3$$

To find the length of the opposite side, we use the Pythagorean Theorem.

$$\begin{align*}
(\text{adjacent})^2 + (\text{opposite})^2 &= (\text{hypotenuse})^2 \\
3^2 + (\text{opposite})^2 &= 5^2 \\
(\text{opposite})^2 &= 25 - 9 = 16 \\
\text{opposite} &= 4
\end{align*}$$

Now that we know the lengths of the three sides, we use the ratios in (1) to find the value of each of the six trigonometric functions:

$$\begin{align*}
sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \\
cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \\
tan \theta &= \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3} \\
csc \theta &= \frac{1}{\sin \theta} = \frac{5}{4} \\
sec \theta &= \frac{1}{\cos \theta} = \frac{5}{3} \\
cot \theta &= \frac{1}{\tan \theta} = \frac{3}{4}
\end{align*}$$

Now that we know the lengths of the three sides, we use the ratios in (1) to find the value of each of the six trigonometric functions:

$$\begin{align*}
sin \theta &= \frac{4}{5} \\
cos \theta &= \frac{3}{5} \\
tan \theta &= \frac{4}{3} \\
csc \theta &= \frac{5}{4} \\
sec \theta &= \frac{5}{3} \\
cot \theta &= \frac{3}{4}
\end{align*}$$

Use the Fundamental Identities

You may have observed some relationships that exist among the six trigonometric functions of acute angles. For example, the reciprocal identities are

$$\begin{align*}
csc \theta &= \frac{1}{\sin \theta} \\
sec \theta &= \frac{1}{\cos \theta} \\
cot \theta &= \frac{1}{\tan \theta}
\end{align*}$$

Two other fundamental identities that are easy to see are the quotient identities.

$$\begin{align*}
tan \theta &= \frac{\sin \theta}{\cos \theta} \\
cot \theta &= \frac{\cos \theta}{\sin \theta}
\end{align*}$$

If $\sin \theta$ and $\cos \theta$ are known, formulas (2) and (3) make it easy to find the values of the remaining trigonometric functions.

Finding the Values of the Remaining Trigonometric Functions, Given $\sin \theta$ and $\cos \theta$

Given $\sin \theta = \frac{\sqrt{5}}{5}$ and $\cos \theta = \frac{2\sqrt{5}}{5}$, find the value of each of the four remaining trigonometric functions of $\theta$.

Solution

Based on formula (3), we have

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{\sqrt{5}}{5}}{\frac{2\sqrt{5}}{5}} = \frac{1}{2}$$
Then we use the reciprocal identities from formula (2) to get

\[
\csc \theta = \frac{1}{\sin \theta} = \frac{1}{\frac{\sqrt{5}}{5}} = \frac{5}{\sqrt{5}} = \sqrt{5} \quad \sec \theta = \frac{1}{\cos \theta} = \frac{1}{\frac{2\sqrt{5}}{5}} = \frac{5}{2\sqrt{5}} = \frac{\sqrt{5}}{2} \quad \cot \theta = \frac{1}{\tan \theta} = \frac{1}{\frac{1}{2}} = 2
\]

**Now Work Problem 21**

Refer now to the right triangle in Figure 22. The Pythagorean Theorem states that \( a^2 + b^2 = c^2 \), which we can write as

\[ b^2 + a^2 = c^2 \]

Dividing each side by \( c^2 \), we get

\[ \frac{b^2}{c^2} + \frac{a^2}{c^2} = 1 \quad \text{or} \quad \left( \frac{b}{c} \right)^2 + \left( \frac{a}{c} \right)^2 = 1 \]

In terms of trigonometric functions of the angle \( \theta \), this equation states that

\[ (\sin \theta)^2 + (\cos \theta)^2 = 1 \]  \hspace{1cm} (4)

Equation (4) is, in fact, an identity, since the equation is true for any acute angle \( \theta \).

It is customary to write \( \sin^2 \theta \) instead of \( (\sin \theta)^2 \), \( \cos^2 \theta \) instead of \( (\cos \theta)^2 \), and so on. With this notation, we can rewrite equation (4) as

\[ \sin^2 \theta + \cos^2 \theta = 1 \]  \hspace{1cm} (5)

Another identity can be obtained from equation (5) by dividing each side by \( \cos^2 \theta \).

\[ \frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \frac{1}{\cos^2 \theta} \]

Now use formulas (2) and (3) to get

\[ \tan^2 \theta + 1 = \sec^2 \theta \]  \hspace{1cm} (6)

Similarly, by dividing each side of equation (5) by \( \sin^2 \theta \), we get \( 1 + \cot^2 \theta = \csc^2 \theta \), which we write as

\[ \cot^2 \theta + 1 = \csc^2 \theta \]  \hspace{1cm} (7)

Collectively, the identities in equations (5), (6), and (7) are referred to as the Pythagorean Identities.

Let’s pause here to summarize the fundamental identities.

### Fundamental Identities

\[
\begin{align*}
\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta} \\
\csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\
\sin^2 \theta + \cos^2 \theta &= 1 & \tan^2 \theta + 1 &= \sec^2 \theta & \cot^2 \theta + 1 &= \csc^2 \theta
\end{align*}
\]
EXAMPLE 3  Finding the Exact Value of a Trigonometric Expression Using Identities

Find the exact value of each expression. Do not use a calculator.

(a) \( \tan 20^\circ = \frac{\sin 20^\circ}{\cos 20^\circ} \)

(b) \( \sin^2 \frac{\pi}{12} + \frac{1}{\sec^2 \frac{\pi}{12}} \)

Solution

(a) \( \tan 20^\circ = \frac{\sin 20^\circ}{\cos 20^\circ} = \tan 20^\circ - \tan 20^\circ = 0 \)

(b) \( \sin^2 \frac{\pi}{12} + \frac{1}{\sec^2 \frac{\pi}{12}} = \sin^2 \frac{\pi}{12} + \cos^2 \frac{\pi}{12} = 1 \)

Now Work  PROBLEM 39

3  Find the Values of the Remaining Trigonometric Functions, Given the Value of One of Them

Once the value of one trigonometric function is known, it is possible to find the value of each of the remaining five trigonometric functions.

EXAMPLE 4  Finding the Values of the Remaining Trigonometric Functions, Given \( \sin \theta, \theta \) Acute

Given that \( \sin \theta = \frac{1}{3} \) and \( \theta \) is an acute angle, find the exact value of each of the remaining five trigonometric functions of \( \theta \).

Solution

We solve this problem in two ways: The first way uses the definition of the trigonometric functions; the second method uses the fundamental identities.

Solution 1  Using the Definition

We draw a right triangle with acute angle \( \theta \), opposite side of length \( b = 1 \), and hypotenuse of length \( c = 3 \) (because \( \sin \theta = \frac{1}{3} = \frac{b}{c} \)). See Figure 23. The adjacent side \( a \) can be found by using the Pythagorean Theorem.

\[
a^2 + 1^2 = 3^2 \quad a^2 + 1 = 9 \quad a^2 = 8 \quad a = 2\sqrt{2}
\]

Now the definitions given in equation (1) can be used to find the value of each of the remaining five trigonometric functions. (Refer back to the method used in Example 1.) Using \( a = 2\sqrt{2}, b = 1, \) and \( c = 3 \), we have

\[
\cos \theta = \frac{a}{c} = \frac{2\sqrt{2}}{3} \quad \tan \theta = \frac{b}{a} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}
\]

\[
\csc \theta = \frac{c}{b} = 3 \quad \sec \theta = \frac{c}{a} = \frac{3\sqrt{2}}{4} \quad \cot \theta = \frac{a}{b} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}
\]
We begin by seeking \( \cos \theta \), which can be found by using the Pythagorean Identity from equation (5).

\[
\sin^2 \theta + \cos^2 \theta = 1 \\
\frac{1}{9} + \cos^2 \theta = 1 \\
\cos^2 \theta = 1 - \frac{1}{9} = \frac{8}{9}
\]

Recall that the trigonometric functions of an acute angle are positive. In particular, \( \cos \theta > 0 \) for an acute angle \( \theta \), so we have

\[
\cos \theta = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}
\]

Now we know that \( \sin \theta = \frac{1}{3} \) and \( \cos \theta = \frac{2\sqrt{2}}{3} \), so we can proceed as we did in Example 2.

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{3}{2\sqrt{2}}}{\frac{2\sqrt{2}}{3}} = \frac{1}{2} = 2
\]

\[
\cot \theta = \frac{1}{\tan \theta} = \frac{1}{\frac{2\sqrt{2}}{4}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}
\]

\[
\sec \theta = \frac{1}{\cos \theta} = \frac{1}{\frac{2\sqrt{2}}{3}} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}
\]

\[
\csc \theta = \frac{1}{\sin \theta} = \frac{1}{\frac{1}{3}} = 3
\]

### Finding the Values of the Trigonometric Functions When One Is Known

Given the value of one trigonometric function of an acute angle \( \theta \), the exact value of each of the remaining five trigonometric functions of \( \theta \) can be found in either of two ways.

**Method 1 Using the Definition**

**Step 1:** Draw a right triangle showing the acute angle \( \theta \).

**Step 2:** Two of the sides can then be assigned values based on the value of the given trigonometric function.

**Step 3:** Find the length of the third side by using the Pythagorean Theorem.

**Step 4:** Use the definitions in equation (1) to find the value of each of the remaining trigonometric functions.

**Method 2 Using Identities**

Use appropriately selected identities to find the value of each of the remaining trigonometric functions.

### Example 5

**Given One Value of a Trigonometric Function, Find the Values of the Remaining Ones**

Given \( \tan \theta = \frac{1}{2} \), \( \theta \) an acute angle, find the exact value of each of the remaining five trigonometric functions of \( \theta \).

**Solution 1 Using the Definition**

Figure 24 shows a right triangle with acute angle \( \theta \), where

\[
\tan \theta = \frac{1}{2} = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}
\]
With $b = 1$ and $a = 2$, the hypotenuse $c$ can be found by using the Pythagorean Theorem.

$$c^2 = a^2 + b^2 = 2^2 + 1^2 = 5$$

$$c = \sqrt{5}$$

Now apply the definitions using $a = 2$, $b = 1$, and $c = \sqrt{5}$.

$$\sin \theta = \frac{b}{c} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\cos \theta = \frac{a}{c} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\csc \theta = \frac{c}{b} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

$$\sec \theta = \frac{c}{a} = \frac{\sqrt{5}}{2}$$

$$\cot \theta = \frac{a}{b} = \frac{2}{1} = 2$$

**Solution 2 Using Identities**

Because we know the value of $\tan \theta$, we use the Pythagorean Identity that involves $\tan \theta$:

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\frac{1}{4} + 1 = \sec^2 \theta$$

$$\sec^2 \theta = \frac{1}{4} + 1 = \frac{5}{4}$$

Proceed to solve for $\sec \theta$.

$$\sec \theta = \frac{\sqrt{5}}{2}$$

Now we know $\tan \theta = \frac{1}{2}$ and $\sec \theta = \frac{\sqrt{5}}{2}$. Using reciprocal identities, we find

$$\cos \theta = \frac{1}{\sec \theta} = \frac{1}{\frac{\sqrt{5}}{2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{1}{\frac{1}{2}} = 2$$

To find $\sin \theta$, we use the following reasoning:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \text{ so } \sin \theta = (\tan \theta)(\cos \theta) = \frac{1}{2} \cdot \frac{2\sqrt{5}}{5} = \frac{\sqrt{5}}{5}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{\frac{\sqrt{5}}{5}} = \sqrt{5}$$

### Now Work Problem 25

**4 Use the Complementary Angle Theorem**

Two acute angles are called **complementary** if their sum is a right angle. Because the sum of the angles of any triangle is $180^\circ$, it follows that, for a right triangle, the two acute angles are complementary.

Refer now to Figure 25; we have labeled the angle opposite side $b$ as $B$ and the angle opposite side $a$ as $A$. Notice that side $a$ is adjacent to angle $B$ and is opposite angle $A$. Similarly, side $b$ is opposite angle $B$ and is adjacent to angle $A$. As a result,

$$\sin B = \frac{b}{c} = \cos A \quad \cos B = \frac{a}{c} = \sin A \quad \tan B = \frac{b}{a} = \cot A$$

$$\csc B = \frac{c}{b} = \sec A \quad \sec B = \frac{c}{a} = \csc A \quad \cot B = \frac{a}{b} = \tan A$$

(8)
Because of these relationships, the functions sine and cosine, tangent and cotangent, and secant and cosecant are called cofunctions of each other. The identities (8) may be expressed in words as follows:

**THEOREM**

**Complementary Angle Theorem**

Cofunctions of complementary angles are equal.

Here are examples of this theorem.

<table>
<thead>
<tr>
<th>Cofunctions</th>
<th>Cofunctions</th>
<th>Cofunctions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin 30^\circ = \cos 60^\circ )</td>
<td>( \tan 40^\circ = \cot 50^\circ )</td>
<td>( \sec 80^\circ = \csc 10^\circ )</td>
</tr>
</tbody>
</table>

If an angle \( \theta \) is measured in degrees, we will use the degree symbol when writing a trigonometric function of \( \theta \), as, for example, in \( \sin 30^\circ \) and \( \tan 45^\circ \). If an angle \( \theta \) is measured in radians, then no symbol is used when writing a trigonometric function of \( \theta \), as, for example, in \( \cos \pi \) and \( \sec \frac{\pi}{3} \).

If \( \theta \) is an acute angle measured in degrees, the angle \( 90^\circ - \theta \) (or \( \frac{\pi}{2} - \theta \), if \( \theta \) is in radians) is the angle complementary to \( \theta \). Table 2 restates the preceding theorem on cofunctions.

<table>
<thead>
<tr>
<th>( \theta ) (Degrees)</th>
<th>( \theta ) (Radians)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta = \cos(90^\circ - \theta) )</td>
<td>( \sin \theta = \cos \left( \frac{\pi}{2} - \theta \right) )</td>
</tr>
<tr>
<td>( \cos \theta = \sin(90^\circ - \theta) )</td>
<td>( \cos \theta = \sin \left( \frac{\pi}{2} - \theta \right) )</td>
</tr>
<tr>
<td>( \tan \theta = \cot(90^\circ - \theta) )</td>
<td>( \tan \theta = \cot \left( \frac{\pi}{2} - \theta \right) )</td>
</tr>
<tr>
<td>( \csc \theta = \sec(90^\circ - \theta) )</td>
<td>( \csc \theta = \sec \left( \frac{\pi}{2} - \theta \right) )</td>
</tr>
<tr>
<td>( \sec \theta = \csc(90^\circ - \theta) )</td>
<td>( \sec \theta = \csc \left( \frac{\pi}{2} - \theta \right) )</td>
</tr>
<tr>
<td>( \cot \theta = \tan(90^\circ - \theta) )</td>
<td>( \cot \theta = \tan \left( \frac{\pi}{2} - \theta \right) )</td>
</tr>
</tbody>
</table>

The angle \( \theta \) in Table 2 is acute. We will see later (Section 8.4) that these results are valid for any angle \( \theta \).

**EXAMPLE 6**

**Using the Complementary Angle Theorem**

(a) \( \sin 62^\circ = \cos(90^\circ - 62^\circ) = \cos 28^\circ \)

(b) \( \tan \frac{\pi}{12} = \cot \left( \frac{\pi}{2} - \frac{\pi}{12} \right) = \cot \frac{5\pi}{12} \)

(c) \( \cos \frac{\pi}{4} = \sin \left( \frac{\pi}{2} - \frac{\pi}{4} \right) = \sin \frac{\pi}{4} \)

(d) \( \csc \frac{\pi}{6} = \sec \left( \frac{\pi}{2} - \frac{\pi}{6} \right) = \sec \frac{\pi}{3} \)
Using the Complementary Angle Theorem

Find the exact value of each expression. Do not use a calculator.

(a) \( \sec 28^\circ - \csc 62^\circ \)  
(b) \( \frac{\sin 35^\circ}{\cos 55^\circ} \)

**Solution**

(a) \[ \sec 28^\circ - \csc 62^\circ = \csc(90^\circ - 28^\circ) - \csc 62^\circ = \csc 62^\circ - \csc 62^\circ = 0 \]

(b) \[ \frac{\sin 35^\circ}{\cos 55^\circ} = \frac{\cos(90^\circ - 35^\circ)}{\cos 55^\circ} = \frac{\cos 55^\circ}{\cos 55^\circ} = 1 \]

Now Work  **Problem 43**

### 7.2 Assess Your Understanding

**‘Are You Prepared?’**  *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. In a right triangle with legs \( a = 6 \) and \( b = 10 \), the Pythagorean Theorem tells us that the hypotenuse \( c = \) _____.

2. The value of the function \( f(x) = 5x - 7 \) at 5 is _____.

**Concepts and Vocabulary**

3. Two acute angles whose sum is a right angle are called _____.

4. The sine and _____ functions are cofunctions.

5. \( \tan 28^\circ = \cot _____ \).

6. For any angle \( \theta \), \( \sin^2 \theta + \cos^2 \theta = _____ \).

7. **True or False**  \( \tan \theta = \frac{\sin \theta}{\cos \theta} \).

8. **True or False**  \( 1 + \tan^2 \theta = \csc^2 \theta \).

9. **True or False**  If \( \theta \) is an acute angle and \( \sec \theta = 3 \), then \( \cos \theta = \frac{1}{3} \).

10. **True or False**  \( \tan \frac{\pi}{5} = \cot \frac{4\pi}{5} \).

### Historical Feature

The name **sine** for the sine function is due to a medieval confusion. The name comes from the Sanskrit word \( jı-va \), (meaning chord), first used in India by Aryabhata the Elder (ad 510). He really meant half-chord, but abbreviated it. This was brought into Arabic as \( jı-ba \), which was meaningless. Because the proper Arabic word \( jaib \) would be written the same way (short vowels are not written out in Arabic), \( jı-ba \), was pronounced as \( jaib \), which meant bosom or hollow, and \( jaib \) remains as the Arabic word for sine to this day. Scholars translating the Arabic works into Latin found that the word **sinus** also meant bosom or hollow, and from **sinus** we get the word **sine**.

The name **tangent**, due to Thomas Finck (1583), can be understood by looking at Figure 26. The line segment \( DC \) is tangent to the circle at \( C \). If then the length of the line segment \( DC \) is

\[ d(O, C) = \frac{d(D, C)}{d(D, C)} = d(O, C) = \tan \alpha \]

The old name for the tangent is **umbra versa** (meaning turned shadow), referring to the use of the tangent in solving height problems with shadows.
CHAPTER 7  Trigonometric Functions

Skill Building

In Problems 11–20, find the value of the six trigonometric functions of the angle \( \theta \) in each figure.

11. \( \theta \)

12. \( \theta \)

13. \( \theta \)

14. \( \theta \)

15. \( \theta \)

16. \( \theta \)

17. \( \theta \)

18. \( \theta \)

19. \( \theta \)

20. \( \theta \)

In Problems 21–24, use identities to find the exact value of each of the four remaining trigonometric functions of the acute angle \( \theta \).

21. \( \sin \theta = \frac{1}{2}, \cos \theta = \frac{\sqrt{3}}{2} \)

22. \( \sin \theta = \frac{\sqrt{3}}{2}, \cos \theta = \frac{1}{2} \)

23. \( \sin \theta = \frac{2}{3}, \cos \theta = \frac{\sqrt{5}}{3} \)

24. \( \sin \theta = \frac{1}{3}, \cos \theta = \frac{2\sqrt{2}}{3} \)

In Problems 25–36, use the definition or identities to find the exact value of each of the remaining five trigonometric functions of the acute angle \( \theta \).

25. \( \sin \theta = \frac{\sqrt{2}}{2} \)

26. \( \cos \theta = \frac{\sqrt{2}}{2} \)

27. \( \cos \theta = \frac{1}{3} \)

28. \( \sin \theta = \frac{\sqrt{3}}{4} \)

29. \( \tan \theta = \frac{1}{2} \)

30. \( \cot \theta = \frac{1}{2} \)

31. \( \sec \theta = 3 \)

32. \( \csc \theta = 5 \)

33. \( \tan \theta = \sqrt{2} \)

34. \( \sec \theta = \frac{5}{3} \)

35. \( \cot \theta = 2 \)

36. \( \cot \theta = 2 \)

In Problems 37–54, use Fundamental Identities and/or the Complementary Angle Theorem to find the exact value of each expression. Do not use a calculator.

37. \( \sin 20^\circ \cdot \cos 20^\circ \)

38. \( \sec^2 28^\circ - \tan^2 28^\circ \)

39. \( \sin 80^\circ \cdot \csc 80^\circ \)

40. \( \tan 10^\circ \cdot \cot 10^\circ \)

41. \( \tan 10^\circ - \frac{\sin 50^\circ}{\cos 50^\circ} \)

42. \( \cot 25^\circ - \frac{\cos 25^\circ}{\sin 25^\circ} \)

43. \( \sin 80^\circ - \cos 52^\circ \)

44. \( \tan 12^\circ - \cot 178^\circ \)

45. \( \cos 10^\circ \cdot \sin 80^\circ \)

46. \( \cos 40^\circ \cdot \sin 50^\circ \)

47. \( 1 - \cos^2 20^\circ - \cos^2 70^\circ \)

48. \( 1 + \tan^2 5^\circ - \csc^2 85^\circ \)

49. \( \tan 20^\circ - \frac{\cos 70^\circ}{\cos 20^\circ} \)

50. \( \cot 40^\circ - \frac{\sin 50^\circ}{\sin 40^\circ} \)

51. \( \tan 35^\circ \cdot \sec 55^\circ \cdot \cos 35^\circ \)

52. \( \cot 25^\circ \cdot \csc 65^\circ \cdot \sin 25^\circ \)

53. \( \cos 35^\circ \cdot \sin 55^\circ + \cos 55^\circ \cdot \sin 35^\circ \)

54. \( \sec 35^\circ \cdot \csc 55^\circ - \tan 35^\circ \cdot \cot 55^\circ \)

55. Given \( \sin 30^\circ = \frac{1}{2} \), use trigonometric identities to find the exact value of:

(a) \( \cos 60^\circ \)

(b) \( \cos^2 30^\circ \)

(c) \( \csc \frac{\pi}{6} \)

(d) \( \sec \frac{\pi}{3} \)

56. Given \( \sin 60^\circ = \frac{\sqrt{3}}{2} \), use trigonometric identities to find the exact value of:

(a) \( \cos 30^\circ \)

(b) \( \cos^2 60^\circ \)

(c) \( \sec \frac{\pi}{6} \)

(d) \( \csc \frac{\pi}{3} \)

57. Given \( \tan \theta = 4 \), use trigonometric identities to find the exact value of:

(a) \( \sec^2 \theta \)

(b) \( \cot \theta \)

(c) \( \cot \left( \frac{\pi}{2} - \theta \right) \)

(d) \( \csc^2 \theta \)

58. Given \( \sec \theta = 3 \), use trigonometric identities to find the exact value of:

(a) \( \cos \theta \)

(b) \( \tan^2 \theta \)

(c) \( \csc (90^\circ - \theta) \)

(d) \( \sin^2 \theta \)

59. Given \( \csc \theta = 4 \), use trigonometric identities to find the exact value of:

(a) \( \sin \theta \)

(b) \( \cot^2 \theta \)

(c) \( \sec (90^\circ - \theta) \)

(d) \( \sec^2 \theta \)

60. Given \( \cot \theta = 2 \), use trigonometric identities to find the exact value of:

(a) \( \tan \theta \)

(b) \( \csc^2 \theta \)

(c) \( \tan \left( \frac{\pi}{2} - \theta \right) \)

(d) \( \sec^2 \theta \)

61. Given the approximation \( \sin 38^\circ \approx 0.62 \), use trigonometric identities to find the approximate value of:

(a) \( \cos 38^\circ \)

(b) \( \tan 38^\circ \)

(c) \( \cot 38^\circ \)

(d) \( \sec 38^\circ \)

(e) \( \csc 38^\circ \)

(f) \( \sin 52^\circ \)

(g) \( \cos 52^\circ \)

(h) \( \tan 52^\circ \)
62. Given the approximation \( \cos 21^\circ = 0.93 \), use trigonometric identities to find the approximate value of
(a) \( \sin 21^\circ \)  
(b) \( \tan 21^\circ \)  
(c) \( \cot 21^\circ \)  
(d) \( \sec 21^\circ \)  
(e) \( \csc 21^\circ \)  
(f) \( \sin 69^\circ \)  
(g) \( \cos 69^\circ \)  
(h) \( \tan 69^\circ \)

63. If \( \sin \theta = 0.3 \), find the exact value of \( \sin \theta + \cos \left( \frac{\pi}{2} - \theta \right) \).

Applications and Extensions

67. Calculating the Time of a Trip  From a parking lot you want to walk to a house on the ocean. The house is located 1500 feet down a paved path that parallels the beach, which is 500 feet wide. Along the path you can walk 300 feet per minute, but in the sand on the beach you can only walk 100 feet per minute. See the illustration.

(a) Calculate the time \( T \) if you walk 1500 feet along the paved path and then 500 feet in the sand to the house.
(b) Calculate the time \( T \) if you walk in the sand directly to the house.
(c) Express the time \( T \) to get from the parking lot to the beachhouse as a function of the angle \( \theta \) shown in the illustration.
(d) Calculate the time \( T \) if you walk directly from the parking lot to the house.
(e) Graph \( T = T(\theta) \). For what angle \( \theta \) is \( T \) least? What is \( x \) for this angle? What is the minimum time?
(f) Explain why \( \tan \theta = \frac{1}{3} \) gives the smallest angle \( \theta \) that is possible.

68. Carrying a Ladder around a Corner  Two hallways, one of width 3 feet, the other of width 4 feet, meet at a right angle. See the illustration.

(a) Express the length \( L \) of the line segment shown as a function of the angle \( \theta \).
(b) Discuss why the length of the longest ladder that can be carried around the corner is equal to the smallest value of \( L \).

69. Electrical Engineering  A resistor and an inductor connected in a series network impede the flow of an alternating current. This impedance \( Z \) is determined by the reactance \( X \) of the inductor and the resistance \( R \) of the resistor. The three quantities, all measured in ohms, can be represented by the sides of a right triangle as illustrated, so \( Z^2 = X^2 + R^2 \). The angle \( \phi \) is called the phase angle. Suppose a series network has an inductive reactance of \( X = 400 \) ohms and a resistance of \( R = 600 \) ohms.

(a) Find the impedance \( Z \).
(b) Find the values of the six trigonometric functions of the phase angle \( \phi \).

70. Electrical Engineering  Refer to Problem 69. A series network has a resistance of \( R = 588 \) ohms. The phase angle \( \phi \) is such that \( \tan \phi = \frac{5}{12} \).

(a) Determine the inductive reactance \( X \) and the impedance \( Z \).
(b) Determine the values of the remaining five trigonometric functions of the phase angle \( \phi \).

71. Geometry  Suppose that the angle \( \theta \) is a central angle of a circle of radius 1 (see the figure). Show that

(a) Angle \( OAC = \frac{\theta}{2} \)
(b) \( |CD| = \sin \theta \) and \( |OD| = \cos \theta \)
(c) \( \tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} \)

72. Geometry  Show that the area \( A \) of an isosceles triangle is \( A = \frac{a^2 \sin \theta \cos \theta}{2} \), where \( a \) is the length of one of the two equal sides and \( \theta \) is the measure of one of the two equal angles (see the figure).

73. Geometry  Let \( n \geq 1 \) be any real number and let \( \theta \) be any angle for which \( 0 < n\theta < \frac{\pi}{2} \). Then we can draw a triangle with the angles \( \theta \) and \( n\theta \) and included side of length 1 (do you see why?) and place it on the unit circle as illustrated.
CHAPTER 7  Trigonometric Functions

Now, drop the perpendicular from C to D = (x, 0) and show that

\[ x = \frac{\tan(n\theta)}{\tan \theta + \tan(n\theta)} \]

74. Geometry  Refer to the figure. The smaller circle, whose radius is a, is tangent to the larger circle, whose radius is b. The ray OA contains a diameter of each circle, and the ray OB is tangent to each circle. Show that

\[ \cos \theta = \frac{\sqrt{ab}}{a + b} \]

(This shows that \( \cos \theta \) equals the ratio of the geometric mean of a and b to the arithmetic mean of a and b.)

[Hint: First show that \( \sin \theta = (b - a)/(b + a) \).]

75. Geometry  Refer to the figure. If |OA| = 1, show that

(a) Area \( \Delta OAC = \frac{1}{2} \sin \alpha \cos \alpha \)

(b) Area \( \Delta OCB = \frac{1}{2} |OB|^2 \sin \beta \cos \beta \)

(c) Area \( \Delta OAB = \frac{1}{2} |OB| \sin(\alpha + \beta) \)

76. Geometry  Refer to the figure, where a unit circle is drawn. The line segment DB is tangent to the circle.

(a) Express the area of \( \Delta OBC \) in terms of \( \sin \theta \) and \( \cos \theta \).

[Hint: Use the altitude from C to the base \( \overline{DB} = 1 \).]

(b) Express the area of \( \Delta OBD \) in terms of \( \sin \theta \) and \( \cos \theta \).

(c) The area of the sector \( OBC \) of the circle is \( \frac{1}{2} \theta \), where \( \theta \) is measured in radians. Use the results of parts (a) and (b) and the fact that

\[ \text{Area } \Delta OBC < \text{Area of sector } OBC < \text{Area } \Delta OBD \]

to show that

\[ 1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta} \]

77. If \( \cos \alpha = \tan \beta \) and \( \cos \beta = \tan \alpha \), where \( \alpha \) and \( \beta \) are acute angles, show that

\[ \sin \alpha = \sin \beta = \frac{\sqrt{3 - \sqrt{5}}}{2} \]

78. If \( \theta \) is an acute angle and \( \tan \theta = x, x \neq 0 \), express the remaining five trigonometric functions in terms of \( x \).

Discussion and Writing

79. If \( \theta \) is an acute angle, explain why see \( \theta > 1 \).

80. If \( \theta \) is an acute angle, explain why \( 0 < \sin \theta < 1 \).

81. How would you explain the meaning of the sine function to a fellow student who has just completed college algebra?

‘Are You Prepared?’ Answers

1. 2\sqrt{34}  
2. \( f(5) = 8 \)

82. Look back at Example 5. Which of the two solutions do you prefer? Explain your reasoning.
In the previous section, we developed ways to find the value of each trigonometric function of an acute angle when one of the functions is known. In this section, we discuss the problem of finding the value of each trigonometric function of an acute angle when the angle is given.

For three special acute angles, we can use some results from plane geometry to find the exact value of each of the six trigonometric functions.

1. Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45^\circ$

**EXAMPLE 1**

Finding the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45^\circ$

Find the exact values of the six trigonometric functions of $\frac{\pi}{4} = 45^\circ$.

Using the right triangle in Figure 27(a), in which one of the angles is $\frac{\pi}{4} = 45^\circ$, it follows that the other acute angle is also $\frac{\pi}{4} = 45^\circ$, and hence the triangle is isosceles. As a result, side $a$ and side $b$ are equal in length. Since the values of the trigonometric functions of an angle depend only on the angle and not on the size of the triangle, we may assign any values to $a$ and $b$ for which $a = b > 0$. We decide to use the triangle for which

$$a = b = 1$$

Then, by the Pythagorean Theorem,

$$c^2 = a^2 + b^2 = 1 + 1 = 2$$

$$c = \sqrt{2}$$

As a result, we have the triangle in Figure 27(b), from which we find

$$\sin \frac{\pi}{4} = \sin 45^\circ = \frac{b}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos \frac{\pi}{4} = \cos 45^\circ = \frac{a}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
CHAPTER 7 Trigonometric Functions

Using Quotient and Reciprocal Identities, we find

\[
\begin{align*}
\tan \frac{\pi}{4} &= \tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = \frac{\sqrt{2}}{\sqrt{2}} = 1 \\
\cot \frac{\pi}{4} &= \cot 45^\circ = \frac{1}{\tan 45^\circ} = \frac{1}{1} = 1 \\
\sec \frac{\pi}{4} &= \sec 45^\circ = \frac{1}{\cos 45^\circ} = \frac{1}{\sqrt{2}} = \sqrt{2} \\
\csc \frac{\pi}{4} &= \csc 45^\circ = \frac{1}{\sin 45^\circ} = \frac{1}{\sqrt{2}} = \sqrt{2} \\
\tan \frac{\pi}{4} &= \tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = \frac{\sqrt{2}}{\sqrt{2}} = 1 \\
\cot \frac{\pi}{4} &= \cot 45^\circ = \frac{1}{\tan 45^\circ} = \frac{1}{1} = 1
\end{align*}
\]

EXAMPLE 2 Finding the Exact Value of a Trigonometric Expression

Find the exact value of each expression.

(a) \((\sin 45^\circ)(\tan 45^\circ)\) \\
(b) \((\sec \frac{\pi}{4})(\cot \frac{\pi}{4})\)

Solution We use the results obtained in Example 1.

(a) \((\sin 45^\circ)(\tan 45^\circ) = \frac{\sqrt{2}}{2} \cdot 1 = \frac{\sqrt{2}}{2}\)

(b) \((\sec \frac{\pi}{4})(\cot \frac{\pi}{4}) = \sqrt{2} \cdot 1 = \sqrt{2}\)

Now Work PROBLEMS 5 AND 17

EXAMPLE 3 Finding the Exact Values of the Trigonometric Functions of \(\frac{\pi}{6} = 30^\circ\) and \(\frac{\pi}{3} = 60^\circ\)

Find the exact values of the six trigonometric functions of \(\frac{\pi}{6} = 30^\circ\) and \(\frac{\pi}{3} = 60^\circ\).

Solution Form a right triangle in which one of the angles is \(\frac{\pi}{6} = 30^\circ\). It then follows that the third angle is \(\frac{\pi}{3} = 60^\circ\). Figure 28(a) illustrates such a triangle with hypotenuse of length 2. Our problem is to determine \(a\) and \(b\).

We begin by placing next to the triangle in Figure 28(a) another triangle congruent to the first, as shown in Figure 28(b). Notice that we now have a triangle whose angles are each 60°. This triangle is therefore equilateral, so each side is of length 2. In particular, the base is \(2a = 2\), so \(a = 1\). By the Pythagorean Theorem, \(b\) satisfies the equation \(a^2 + b^2 = c^2\), so we have

\[
\begin{align*}
a^2 + b^2 &= c^2 \\
1^2 + b^2 &= 2^2 \\
b^2 &= 4 - 1 = 3 \\
b &= \sqrt{3}
\end{align*}
\]
Using the triangle in Figure 28(c) and the fact that \( \frac{\pi}{6} = 30° \) and \( \frac{\pi}{3} = 60° \) are complementary angles, we find:

\[
\begin{align*}
\sin \frac{\pi}{6} &= \sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \\
\cos \frac{\pi}{6} &= \cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \\
\sin \frac{\pi}{3} &= \sin 60° = \frac{\sqrt{3}}{2} \\
\cos \frac{\pi}{3} &= \cos 60° = \frac{1}{2} \\
\tan \frac{\pi}{6} &= \tan 30° = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \\
\cot \frac{\pi}{6} &= \cot 30° = \frac{\sqrt{3}}{1} = \sqrt{3} \\
\csc \frac{\pi}{6} &= \csc 30° = \frac{2}{1} = 2 \\
\sec \frac{\pi}{6} &= \sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\
\csc \frac{\pi}{3} &= \csc 60° = \sqrt{3} \\
\sec \frac{\pi}{3} &= \sec 60° = 2 \\
\tan \frac{\pi}{3} &= \tan 60° = \sqrt{3}
\end{align*}
\]

Table 3 summarizes the information just derived for the angles \( \frac{\pi}{6} = 30°, \frac{\pi}{4} = 45°, \) and \( \frac{\pi}{3} = 60°. \) Rather than memorize the entries in Table 3, you can draw the appropriate triangle to determine the values given in the table.

<table>
<thead>
<tr>
<th>( \theta ) (Radians)</th>
<th>( \theta ) (Degrees)</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \csc \theta )</th>
<th>( \sec \theta )</th>
<th>( \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\pi}{6} )</td>
<td>30°</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>2</td>
<td>( \frac{2\sqrt{3}}{3} )</td>
<td>( \sqrt{3} )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>45°</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>1</td>
<td>( \sqrt{2} )</td>
<td>( \sqrt{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
<td>60°</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>( \frac{2\sqrt{3}}{3} )</td>
<td>2</td>
<td>( \frac{\sqrt{3}}{3} )</td>
</tr>
</tbody>
</table>

**Example 4**

Finding the Exact Value of a Trigonometric Expression

Find the exact value of each expression.

(a) \( \sin 45° \cos 30° \) \hspace{1cm} (b) \( \tan \frac{\pi}{4} - \sin \frac{\pi}{3} \) \hspace{1cm} (c) \( \tan^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{4} \)

**Solution**

(a) \( \sin 45° \cos 30° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{4} \)

(b) \( \tan \frac{\pi}{4} - \sin \frac{\pi}{3} = 1 - \frac{\sqrt{3}}{2} = \frac{2 - \sqrt{3}}{2} \)

(c) \( \tan^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{4} = \left( \frac{\sqrt{3}}{2} \right)^2 + \left( \frac{\sqrt{2}}{2} \right)^2 = \frac{3}{4} + \frac{1}{2} = \frac{5}{6} \)
The exact values of the trigonometric functions for the angles $\frac{\pi}{6} = 30^\circ$, $\frac{\pi}{4} = 45^\circ$, and $\frac{\pi}{3} = 60^\circ$ are relatively easy to calculate, because the triangles that contain such angles have “nice” geometric features. For most other angles, we can only approximate the value of each trigonometric function. To do this, we will need a calculator.

### 3 Use a Calculator to Approximate the Values of the Trigonometric Functions of Acute Angles

Before getting started, you must first decide whether to enter the angle in the calculator using radians or degrees and then set the calculator to the correct MODE. (Check your instruction manual to find out how your calculator handles degrees and radians.) Your calculator has the keys marked $\sin$, $\cos$, and $\tan$. To find the values of the remaining three trigonometric functions (secant, cosecant, and cotangent), we use the reciprocal identities:

\[
\begin{align*}
\sec \theta &= \frac{1}{\cos \theta} \\
\csc \theta &= \frac{1}{\sin \theta} \\
\cot \theta &= \frac{1}{\tan \theta}
\end{align*}
\]

**EXAMPLE 5**

**Using a Calculator to Approximate the Value of Trigonometric Functions**

Use a calculator to find the approximate value of:

(a) $\cos 48^\circ$  
(b) $\csc 21^\circ$  
(c) $\tan \frac{\pi}{12}$

Express your answer rounded to two decimal places.

**Solution**

(a) First, we set the MODE to receive degrees. Rounded to two decimal places, $\cos 48^\circ = 0.67$

(b) Most calculators do not have a csc key. The manufacturers assume the user knows some trigonometry. To find the value of $\csc 21^\circ$, we use the fact that $\csc 21^\circ = \frac{1}{\sin 21^\circ}$. Rounded to two decimal places, $\csc 21^\circ = 2.79$.

(c) Set the MODE to receive radians. Figure 29 shows the solution using a TI-84 Plus graphing calculator. Rounded to two decimal places, $\tan \frac{\pi}{12} = 0.27$

### 4 Model and Solve Applied Problems Involving Right Triangles

Right triangles can be used to model many types of situations, such as the optimal design of a rain gutter.*

* In applied problems, it is important that answers be reported with both justifiable accuracy and appropriate significant figures. We shall assume that the problem data are accurate to the number of significant digits, resulting in sides being rounded to two decimal places and angles being rounded to one decimal place.
Constructing a Rain Gutter

A rain gutter is to be constructed of aluminum sheets 12 inches wide. After marking off a length of 4 inches from each edge, this length is bent up at an angle \( \theta \). See Figure 30.

(a) Express the area \( A \) of the opening as a function of \( \theta \).

[Hint: Let \( b \) denote the vertical height of the bend.]

(b) Find the area \( A \) of the opening for \( \theta = 30^\circ \), \( \theta = 45^\circ \), \( \theta = 60^\circ \), and \( \theta = 75^\circ \).

(c) Graph \( A = A(\theta) \). Find the angle \( \theta \) that makes \( A \) largest. (This bend will allow the most water to flow through the gutter.)

Solution

(a) Look again at Figure 30. The area \( A \) of the opening is the sum of the areas of two congruent right triangles and one rectangle. Look at Figure 31, showing one of the triangles in Figure 30 redrawn. We see that

\[
\cos \theta = \frac{a}{4} \quad \text{so} \quad a = 4 \cos \theta \quad \sin \theta = \frac{b}{4} \quad \text{so} \quad b = 4 \sin \theta
\]

The area of the triangle is

\[
\text{area of triangle} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}ab = \frac{1}{2}(4 \cos \theta)(4 \sin \theta) = 8 \sin \theta \cos \theta
\]

So the area of the two congruent triangles is \( 16 \sin \theta \cos \theta \).

The rectangle has length 4 and height \( b \), so its area is

\[
\text{area of rectangle} = 4b = 4(4 \sin \theta) = 16 \sin \theta
\]

The area \( A \) of the opening is

\[
A = \text{area of the two triangles} + \text{area of the rectangle}
\]

\[
A(\theta) = 16 \sin \theta \cos \theta + 16 \sin \theta = 16 \sin \theta(\cos \theta + 1)
\]

(b) For \( \theta = 30^\circ \):

\[
A(30^\circ) = 16 \sin 30^\circ(\cos 30^\circ + 1)
\]

\[
= 16 \left( \frac{1}{2} \right) \left( \frac{\sqrt{3}}{2} + 1 \right) = 4\sqrt{3} + 8 \approx 14.9
\]

The area of the opening for \( \theta = 30^\circ \) is about 14.9 square inches.

For \( \theta = 45^\circ \):

\[
A(45^\circ) = 16 \sin 45^\circ(\cos 45^\circ + 1)
\]

\[
= 16 \left( \frac{\sqrt{2}}{2} \right) \left( \frac{\sqrt{2}}{2} + 1 \right) = 8 + 8\sqrt{2} \approx 19.3
\]

The area of the opening for \( \theta = 45^\circ \) is about 19.3 square inches.

For \( \theta = 60^\circ \):

\[
A(60^\circ) = 16 \sin 60^\circ(\cos 60^\circ + 1)
\]

\[
= 16 \left( \frac{\sqrt{3}}{2} \right) \left( \frac{1}{2} + 1 \right) = 12\sqrt{3} \approx 20.8
\]

The area of the opening for \( \theta = 60^\circ \) is about 20.8 square inches.

For \( \theta = 75^\circ \):

\[
A(75^\circ) = 16 \sin 75^\circ(\cos 75^\circ + 1) \approx 19.5
\]

The area of the opening for \( \theta = 75^\circ \) is about 19.5 square inches.

(c) Figure 32 shows the graph of \( A = A(\theta) \). Using MAXIMUM, the angle \( \theta \) that makes \( A \) largest is \( 60^\circ \).
In addition to developing models using right triangles, we can use right triangle trigonometry to measure heights and distances that are either awkward or impossible to measure by ordinary means. When using right triangles to solve these problems, pay attention to the known measures. This will indicate what trigonometric function to use. For example, if we know the measure of an angle and the length of the side adjacent to the angle, and wish to find the length of the opposite side, we would use the tangent function. Do you know why?

**EXAMPLE 7**

**Finding the Width of a River**

A surveyor can measure the width of a river by setting up a transit* at a point C on one side of the river and taking a sighting of a point A on the other side. Refer to Figure 33. After turning through an angle of 90° at C, the surveyor walks a distance of 200 meters to point B. Using the transit at B, the angle \( \theta \) is measured and found to be 20°. What is the width of the river rounded to the nearest meter?

**Solution**

We seek the length of side \( b \). We know \( a \) and \( \theta \). So we use the fact that \( b \) is opposite \( \theta \) and \( a \) is adjacent to \( \theta \) and write

\[
\tan \theta = \frac{b}{a}
\]

which leads to

\[
\tan 20° = \frac{b}{200}
\]

\[
b = 200 \tan 20° \approx 72.79 \text{ meters}
\]

The width of the river is 73 meters, rounded to the nearest meter.

**Now Work**

**PROBLEM 59**

Vertical heights can sometimes be measured using either the angle of elevation or the angle of depression. If a person is looking up at an object, the acute angle measured from the horizontal to a line of sight to the object is called the **angle of elevation**. See Figure 34(a).

---

* An instrument used in surveying to measure angles.
If a person is standing on a cliff looking down at an object, the acute angle made by the line of sight to the object and the horizontal is called the **angle of depression**. See Figure 34(b).

### Example 8 Finding the Height of a Cloud

Meteorologists find the height of a cloud using an instrument called a **ceilometer**. A ceilometer consists of a light projector that directs a vertical light beam up to the cloud base and a light detector that scans the cloud to detect the light beam. See Figure 35(a). On December 1, 2006, at Midway Airport in Chicago, a ceilometer was employed to find the height of the cloud cover. It was set up with its light detector 300 feet from its light projector. If the angle of elevation from the light detector to the base of the cloud is 75°, what is the height of the cloud cover?

![Figure 35](image)

**Solution** Figure 35(b) illustrates the situation. To find the height \( h \), we use the fact that \( \tan 75° = \frac{h}{300} \) so

\[
    h = 300 \tan 75° \approx 1120 \text{ feet}
\]

The ceiling (height to the base of the cloud cover) is approximately 1120 feet.

Now Work **Problem 61**

The idea behind Example 8 can also be used to find the height of an object with a base that is not accessible to the horizontal.

### Example 9 Finding the Height of a Statue on a Building

Adorning the top of the Board of Trade building in Chicago is a statue of Ceres, the Roman goddess of wheat. From street level, two observations are taken 400 feet from the center of the building. The angle of elevation to the base of the statue is found to be 55.1° and the angle of elevation to the top of the statue is 56.5°. See Figure 36(a). What is the height of the statue?
CHAPTER 7  Trigonometric Functions

Solution  Figure 36(b) shows two triangles that replicate Figure 36(a). The height of the statue of Ceres will be \( b' - b \). To find \( b \) and \( b' \), we refer to Figure 36(b).

\[
\begin{align*}
\tan 55.1^\circ &= \frac{b}{400} & \tan 56.5^\circ &= \frac{b'}{400} \\
 b &= 400 \tan 55.1^\circ \approx 573.39 & b' &= 400 \tan 56.5^\circ \approx 604.33
\end{align*}
\]

The height of the statue is approximately \( 604.33 - 573.39 = 30.94 \) feet.

7.3 Assess Your Understanding

Concepts and Vocabulary

1. \( \tan \frac{\pi}{4} + \sin 30^\circ = \ldots \).
2. Using a calculator, \( \sin 2 = \ldots \), rounded to two decimal places.

Skill Building

5. Write down the exact value of each of the six trigonometric functions of 45°.

In Problems 7–16, find the exact value of each expression if \( u = 60° \).

7. \( f(u) \)
8. \( g(u) \)
9. \( f\left(\frac{\theta}{2}\right) \)
10. \( g\left(\frac{\theta}{2}\right) \)
11. \( f(u)^2 \)
12. \( [g(\theta)]^2 \)
13. \( 2f(\theta) \)
14. \( 2g(\theta) \)
15. \( \frac{f(\theta)}{2} \)
16. \( \frac{g(\theta)}{2} \)

In Problems 17–28, find the exact value of each expression. Do not use a calculator.

17. \( 4 \cos 45^\circ - 2 \sin 45^\circ \)
18. \( 2 \sin 45^\circ + 4 \cos 30^\circ \)
19. \( 6 \tan 45^\circ - 8 \cos 60^\circ \)
20. \( \sin 30^\circ \cdot \tan 60^\circ \)
21. \( \sec \frac{\pi}{3} + 2 \csc \frac{\pi}{3} \)
22. \( \tan \frac{\pi}{4} + \cot \frac{\pi}{4} \)
23. \( \sec^2 \frac{\pi}{6} - 4 \)
24. \( 4 + \tan^2 \frac{\pi}{3} \)
25. \( \sin^2 30^\circ + \cos^2 60^\circ \)
26. \( \sec^2 60^\circ - \tan^2 45^\circ \)
27. \( 1 - \cos^2 30^\circ - \cos^2 60^\circ \)
28. \( 1 + \tan^2 30^\circ - \csc^2 45^\circ \)

In Problems 29–46, use a calculator to find the approximate value of each expression. Round the answer to two decimal places.

29. \( \sin 28^\circ \)
30. \( \cos 14^\circ \)
31. \( \tan 21^\circ \)
32. \( \cot 70^\circ \)
33. \( \sec 41^\circ \)
34. \( \csc 55^\circ \)
35. \( \sin \frac{\pi}{10} \)
36. \( \cos \frac{\pi}{8} \)
37. \( \tan \frac{5\pi}{12} \)
38. \( \cot \frac{\pi}{18} \)
39. \( \sec \frac{\pi}{12} \)
40. \( \csc \frac{5\pi}{13} \)
41. \( \sin 1^\circ \)
42. \( \tan 1^\circ \)
43. \( \sin 1^\circ \)
44. \( \tan 1^\circ \)
45. \( \tan 0.3 \)
46. \( \tan 0.1 \)

Applications and Extensions

Problems 47–51 require the following discussion.

Projectile Motion  The path of a projectile fired at an inclination \( \theta \) to the horizontal with initial speed \( v_0 \) is a parabola. See the figure. The range \( R \) of the projectile, that is, the horizontal distance that the projectile travels, is found by using the function

\[
R(\theta) = \frac{2v_0^2 \sin \theta \cos \theta}{g}
\]

where \( g \approx 32.2 \) feet per second per second \( \approx 9.8 \) meters per second per second is the acceleration due to gravity. The maximum height \( H \) of the projectile is given by the function

\[
H(\theta) = \frac{v_0^2 \sin^2 \theta}{2g}
\]
SECTION 7.3  Computing the Values of Trigonometric Functions of Acute Angles

In Problems 47–50, find the range \( R \) and maximum height \( H \) of the projectile. Round answers to two decimal places.

47. The projectile is fired at an angle of 45° to the horizontal with an initial speed of 100 feet per second.

48. The projectile is fired at an angle of 30° to the horizontal with an initial speed of 150 meters per second.

49. The projectile is fired at an angle of 25° to the horizontal with an initial speed of 500 meters per second.

50. The projectile is fired at an angle of 50° to the horizontal with an initial speed of 200 feet per second.

51. Inclined Plane  See the illustration. If friction is ignored, the time \( t \) (in seconds) required for a block to slide down an inclined plane is given by the function

\[
t(\theta) = \sqrt{\frac{2a}{g \sin \theta \cos \theta}}
\]

where \( a \) is the length (in feet) of the base and \( g \approx 32 \) feet per second per second is the acceleration due to gravity. How long does it take a block to slide down an inclined plane with base \( a = 10 \) feet when

(a) \( \theta = 30° \)  (b) \( \theta = 45° \)  (c) \( \theta = 60° \)?

52. Piston Engines  See the illustration. In a certain piston engine, the distance \( x \) (in inches) from the center of the drive shaft to the head of the piston is given by the function

\[
x(\theta) = \cos \theta + \sqrt{\frac{16 + 0.5(2 \cos^2 \theta - 1)}{}}
\]

where \( \theta \) is the angle between the crank and the path of the piston head. Find \( x \) when \( \theta = 30° \) and when \( \theta = 45° \).

53. Calculating the Time of a Trip  Two oceanfront homes are located 8 miles apart on a straight stretch of beach, each a distance of 1 mile from a paved road that parallels the ocean. Sally can jog 8 miles per hour along the paved road, but only 3 miles per hour in the sand on the beach. Because a river flows between the two houses, it is necessary to jog on the sand to the road, continue on the road, and then jog on the sand to get from one house to the other. See the illustration.

(a) Express the time \( T \) to get from one house to the other as a function of the angle shown in the illustration.

(b) Calculate the time \( T \) for How long is Sally on the paved road?

(c) Calculate the time \( T \) for How long is Sally on the paved road?

(d) Calculate the time \( T \) for How long is Sally on the paved road?

(e) Calculate the time \( T \) for Describe the path taken. Explain why \( \theta \) must be larger than 14°.

(f) Graph \( T = T(\theta) \). What angle \( \theta \) results in the least time? What is the least time? How long is Sally on the paved road?

54. Designing Fine Decorative Pieces  A designer of decorative art plans to market solid gold spheres encased in clear crystal cones. Each sphere is of fixed radius \( R \) and will be enclosed in a cone of height \( h \) and radius \( r \). See the illustration. Many cones can be used to enclose the sphere, each having a different slant angle \( \theta \).

(a) Express the volume \( V \) of the cone as a function of the slant angle \( \theta \) of the cone.

(b) What volume \( V \) is required to enclose a sphere of radius 2 centimeters in a cone whose slant angle \( \theta \) is 30°? 45°? 60°?

(c) What slant angle \( \theta \) should be used for the volume \( V \) of the cone to be a minimum? (This choice minimizes the amount of crystal required and gives maximum emphasis to the gold sphere.)

55. Geometry  A right triangle has a hypotenuse of length 8 inches. If one angle is 35°, find the length of each leg.

56. Geometry  A right triangle has a hypotenuse of length 10 centimeters. If one angle is 40°, find the length of each leg.
57. Geometry A right triangle contains a $25^\circ$ angle.
   (a) If one leg is of length 5 inches, what is the length of the hypotenuse?
   (b) There are two answers. How is this possible?

58. Geometry A right triangle contains an angle of $\frac{\pi}{8}$ radian.
   (a) If one leg is of length 3 meters, what is the length of the hypotenuse?
   (b) There are two answers. How is this possible?

59. Finding the Width of a Gorge Find the distance from $A$ to $C$ across the gorge illustrated in the figure.

60. Finding the Distance across a Pond Find the distance from $A$ to $C$ across the pond illustrated in the figure.

61. The Eiffel Tower The tallest tower built before the era of television masts, the Eiffel Tower was completed on March 31, 1889. Find the height of the Eiffel Tower (before a television mast was added to the top) using the information given in the illustration.

62. Finding the Distance of a Ship from Shore A person in a small boat, offshore from a vertical cliff known to be 100 feet in height, takes a sighting of the top of the cliff. If the angle of elevation is found to be $25^\circ$, how far offshore is the ship?

63. Finding the Distance to a Plateau Suppose that you are headed toward a plateau 50 meters high. If the angle of elevation to the top of the plateau is $20^\circ$, how far are you from the base of the plateau?

64. Finding the Reach of a Ladder A 22-foot extension ladder leaning against a building makes a $70^\circ$ angle with the ground. How far up the building does the ladder touch?

65. Finding the Distance between Two Objects A blimp, suspended in the air at a height of 500 feet, lies directly over a line from Soldier Field to the Adler Planetarium on Lake Michigan (see the figure). If the angle of depression from the blimp to the stadium is $32^\circ$ and from the blimp to the planetarium is $23^\circ$, find the distance between Soldier Field and the Adler Planetarium.

66. Hot-air Balloon While taking a ride in a hot-air balloon in Napa Valley, Francisco wonders how high he is. To find out, he chooses a landmark that is to the east of the balloon and measures the angle of depression to be $54^\circ$. A few minutes later, after traveling 100 feet east, the angle of depression to the same landmark is determined to be $61^\circ$. Use this information to determine the height of the balloon.

67. Mt. Rushmore To measure the height of Lincoln’s caricature on Mt. Rushmore, two sightings 800 feet from the base of the mountain are taken. If the angle of elevation to the bottom of Lincoln’s face is $32^\circ$ and the angle of elevation to the top is $35^\circ$, what is the height of Lincoln’s face?

68. The CN Tower The CN Tower, located in Toronto, Canada, is the tallest structure in the world. While visiting Toronto, a tourist wondered what the height of the tower above the top of the Sky Pod is. While standing 4000 feet from the tower, she measured the angle to the top of the Sky Pod to be $20.1^\circ$. At this same distance, the angle of elevation to the top of the tower was found to be $24.4^\circ$. Use this information to determine the height of the tower above the Sky Pod.

69. Finding the Length of a Guy Wire A radio transmission tower is 200 feet high. How long should a guy wire be if it is to be attached to the tower 10 feet from the top and is to make an angle of $69^\circ$ with the ground?

70. Finding the Height of a Tower A guy wire 80 feet long is attached to the top of a radio transmission tower, making an angle of $65^\circ$ with the ground. How high is the tower?

71. Washington Monument The angle of elevation of the Sun is $35.1^\circ$ at the instant the shadow cast by the Washington
Monument is 789 feet long. Use this information to calculate the height of the monument.

72. **Finding the Length of a Mountain Trail**  A straight trail with an inclination of 17° leads from a hotel at an elevation of 9000 feet to a mountain lake at an elevation of 11,200 feet. What is the length of the trail?

73. **Constructing a Highway**  A highway whose primary directions are north–south is being constructed along the west coast of Florida. Near Naples, a bay obstructs the straight path of the road. Since the cost of a bridge is prohibitive, engineers decide to go around the bay. The illustration shows the path that they decide on and the measurements taken. What is the length of highway needed to go around the bay?

74. **Photography**  A camera is mounted on a tripod 4 feet high at a distance of 10 feet from George, who is 6 feet tall. See the illustration. If the camera lens has angles of depression and elevation of 20°, will George’s feet and head be seen by the lens? If not, how far back will the camera need to be moved to include George’s feet and head?

75. **Calculating Pool Shots**  A Pool player located at X wants to shoot the white ball off the top cushion and hit the red ball dead center. He knows from physics that the white ball will come off a cushion at the same angle as it hits a cushion. Where on the top cushion should he hit the white ball?

76. **The Freedom Tower**  The Freedom Tower is to be the centerpiece of the rebuilding of the World Trade Center in New York City. The tower will be 1776 feet tall (not including a broadcast antenna). The angle of elevation from the base of an office building to the top of the tower is 34°. The angle of elevation from the helipad on the roof of the office building to the top of the tower is 20°.

(a) How far away is the office building from the Freedom Tower? Assume the side of the tower is vertical. Round to the nearest foot.
(b) How tall is the office building? Round to the nearest foot.

77. Use a calculator set in radian mode to complete the following table. What can you conclude about the value of f(θ) = \( \frac{\sin \theta}{\theta} \) as θ approaches 0?

<table>
<thead>
<tr>
<th>θ</th>
<th>0.5</th>
<th>0.4</th>
<th>0.2</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0001</th>
<th>0.00001</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(θ) = ( \frac{\sin \theta}{\theta} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

78. Use a calculator set in radian mode to complete the following table. What can you conclude about the value of g(θ) = \( \frac{\cos \theta - 1}{\theta} \) as θ approaches 0?

<table>
<thead>
<tr>
<th>θ</th>
<th>0.5</th>
<th>0.4</th>
<th>0.2</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0001</th>
<th>0.00001</th>
</tr>
</thead>
<tbody>
<tr>
<td>g(θ) = ( \frac{\cos \theta - 1}{\theta} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

79. Find the exact value of tan 1° · tan 2° · tan 3° · . . . · tan 89°.
80. Find the exact value of cot 1° · cot 2° · cot 3° · . . . · cot 89°.
81. Find the exact value of cos 1° · cos 2° · . . . · cos 45° · csc 46° · . . . · csc 89°.
82. Find the exact value of sin 1° · sin 2° · . . . · sin 45° · sec 46° · . . . · sec 89°.
7.4 Trigonometric Functions of General Angles

OBJECTIVES
1. Find the Exact Values of the Trigonometric Functions for General Angles (p. 540)
2. Use Coterminal Angles to Find the Exact Value of a Trigonometric Function (p. 542)
3. Determine the Signs of the Trigonometric Functions of an Angle in a Given Quadrant (p. 544)
4. Find the Reference Angle of a General Angle (p. 545)
5. Use a Reference Angle to Find the Exact Value of a Trigonometric Function (p. 546)
6. Find the Exact Values of Trigonometric Functions of an Angle, Given Information about the Functions (p. 547)

Find the Exact Values of the Trigonometric Functions for General Angles

To extend the definition of the trigonometric functions to include angles that are not acute, we employ a rectangular coordinate system and place the angle in the standard position so that its vertex is at the origin and its initial side is along the positive x-axis. See Figure 37.

DEFINITION
Let \( \theta \) be any angle in standard position, and let \((a, b)\) denote the coordinates of any point, except the origin \((0, 0)\), on the terminal side of \( \theta \). If \( r = \sqrt{a^2 + b^2} \) denotes the distance from \((0, 0)\) to \((a, b)\), then the six trigonometric functions of \( \theta \) are defined as the ratios

\[
\begin{align*}
\sin \theta &= \frac{b}{r} \\
\cos \theta &= \frac{a}{r} \\
\tan \theta &= \frac{b}{a} \\
\csc \theta &= \frac{r}{b} \\
\sec \theta &= \frac{r}{a} \\
\cot \theta &= \frac{a}{b}
\end{align*}
\]

provided no denominator equals 0. If a denominator equals 0, that trigonometric function of the angle \( \theta \) is not defined.

Notice in the preceding definitions that if \( a = 0 \), that is, if the point \((a, b)\) is on the y-axis, then the tangent function and the secant function are undefined. Also, if \( b = 0 \), that is, if the point \((a, b)\) is on the x-axis, then the cosecant function and the cotangent function are undefined.

By constructing similar triangles, you should be convinced that the values of the six trigonometric functions of an angle \( \theta \) do not depend on the selection of the point \((a, b)\) on the terminal side of \( \theta \), but rather depend only on the angle \( \theta \) itself. See Figure 38 for an illustration of this when \( \theta \) lies in quadrant II. Since the triangles are similar, the ratio \( \frac{b}{r} \) equals the ratio \( \frac{b'}{r'} \), which equals \( \sin \theta \). Also, the ratio \( \frac{|a|}{r} \) equals the ratio \( \frac{|a'|}{r'} \), so \( \frac{a}{r} = \frac{a'}{r'} \), which equals \( \cos \theta \). And so on.
Also, observe that if $\theta$ is acute these definitions reduce to the right triangle definitions given in Section 7.2, as illustrated in Figure 39.

Finally, from the definition of the six trigonometric functions of a general angle, we see that the Quotient and Reciprocal Identities hold. Also, using $r^2 = a^2 + b^2$ and dividing each side by $r^2$, we can derive the Pythagorean Identities for general angles.

**EXAMPLE 1**

**Finding the Exact Values of the Six Trigonometric Functions of $\theta$, Given a Point on the Terminal Side**

Find the exact value of each of the six trigonometric functions of a positive angle $\theta$ if $(4, -3)$ is a point on its terminal side.

Figure 40 illustrates the situation. For the point $(a, b) = (4, -3)$, we have $a = 4$ and $b = -3$. Then $r = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$, so that

\[
\sin \theta = \frac{b}{r} = -\frac{3}{5}, \quad \cos \theta = \frac{a}{r} = \frac{4}{5}, \quad \tan \theta = \frac{b}{a} = -\frac{3}{4},
\]

\[
\csc \theta = \frac{r}{b} = -\frac{5}{3}, \quad \sec \theta = \frac{r}{a} = \frac{5}{4}, \quad \cot \theta = \frac{a}{b} = -\frac{4}{3}.
\]

In the next example, we find the exact value of each of the six trigonometric functions at the quadrantal angles $0^\circ$, $\frac{\pi}{2}$, $\pi$, and $\frac{3\pi}{2}$.

**EXAMPLE 2**

**Finding the Exact Values of the Six Trigonometric Functions of Quadrantal Angles**

Find the exact values of each of the six trigonometric functions of

(a) $\theta = 0^\circ$  (b) $\theta = \frac{\pi}{2} = 90^\circ$  (c) $\theta = \pi = 180^\circ$  (d) $\theta = \frac{3\pi}{2} = 270^\circ$

(a) The point $P = (1, 0)$ is on the terminal side of $\theta = 0^\circ$ and is a distance of 1 unit from the origin. See Figure 41. Then

\[
\sin 0 = \sin 0^\circ = \frac{0}{1} = 0, \quad \cos 0 = \cos 0^\circ = \frac{1}{1} = 1
\]

\[
\tan 0 = \tan 0^\circ = \frac{0}{1} = 0, \quad \sec 0 = \sec 0^\circ = \frac{1}{1} = 1
\]

Since the $y$-coordinate of $P$ is 0, $\csc 0$ and $\cot 0$ are not defined.

(b) The point $P = (0, 1)$ is on the terminal side of $\theta = \frac{\pi}{2} = 90^\circ$ and is a distance of 1 unit from the origin. See Figure 42. Then

\[
\sin \frac{\pi}{2} = \sin 90^\circ = \frac{1}{1} = 1, \quad \cos \frac{\pi}{2} = \cos 90^\circ = \frac{0}{1} = 0
\]

\[
\csc \frac{\pi}{2} = \csc 90^\circ = \frac{1}{1} = 1, \quad \cot \frac{\pi}{2} = \cot 90^\circ = \frac{0}{1} = 0
\]

Since the $x$-coordinate of $P$ is 0, $\tan \frac{\pi}{2}$ and $\sec \frac{\pi}{2}$ are not defined.
(c) The point \( P = (-1, 0) \) is on the terminal side of \( \theta = \pi = 180^\circ \) and is a distance of 1 unit from the origin. See Figure 43. Then

\[
\begin{align*}
\sin \pi &= \sin 180^\circ = 0 \\
\cos \pi &= \cos 180^\circ = -1 \\
\tan \pi &= \tan 180^\circ = 0 \\
\sec \pi &= \sec 180^\circ = -1
\end{align*}
\]

Since the \( y \)-coordinate of \( P \) is 0, \( \csc \pi \) and \( \cot \pi \) are not defined.

(d) The point \( P = (0, -1) \) is on the terminal side of \( \theta = \frac{3\pi}{2} = 270^\circ \) and is a distance of 1 unit from the origin. See Figure 44. Then

\[
\begin{align*}
\sin \frac{3\pi}{2} &= \sin 270^\circ = -1 \\
\cos \frac{3\pi}{2} &= \cos 270^\circ = 0 \\
\csc \frac{3\pi}{2} &= \csc 270^\circ = -1 \\
\cot \frac{3\pi}{2} &= \cot 270^\circ = 0
\end{align*}
\]

Since the \( x \)-coordinate of \( P \) is 0, \( \tan \frac{3\pi}{2} \) and \( \sec \frac{3\pi}{2} \) are not defined.

Table 4 summarizes the values of the trigonometric functions found in Example 2.

<table>
<thead>
<tr>
<th>( \theta ) (Radians)</th>
<th>( \theta ) (Degrees)</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \csc \theta )</th>
<th>( \sec \theta )</th>
<th>( \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0^\circ</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>Not defined</td>
<td>1</td>
<td>Not defined</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>90^\circ</td>
<td>1</td>
<td>0</td>
<td>Not defined</td>
<td>1</td>
<td>Not defined</td>
<td>0</td>
</tr>
<tr>
<td>( \pi )</td>
<td>180^\circ</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>Not defined</td>
<td>-1</td>
<td>Not defined</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>270^\circ</td>
<td>-1</td>
<td>0</td>
<td>Not defined</td>
<td>-1</td>
<td>Not defined</td>
<td>0</td>
</tr>
</tbody>
</table>

There is no need to memorize Table 4. To find the value of a trigonometric function of a quadrantal angle, draw the angle and apply the definition, as we did in Example 2.

### 2) Use Coterminal Angles to Find the Exact Value of a Trigonometric Function

**DEFINITION**

Two angles in standard position are said to be **coterminal** if they have the same terminal side.

See Figure 45.

For example, the angles 60° and 420° are coterminal, as are the angles −40° and 320°.
In general, if $\theta$ is an angle measured in degrees, then $\theta + 360^\circ k$, where $k$ is any integer, is an angle coterminal with $\theta$. If $\theta$ is measured in radians, then $\theta + 2\pi k$, where $k$ is any integer, is an angle coterminal with $\theta$.

Because coterminal angles have the same terminal side, it follows that the values of the trigonometric functions of coterminal angles are equal. We use this fact in the next example.

### Example 3

**Using a Coterminal Angle to Find the Exact Value of a Trigonometric Function**

Find the exact value of each of the following:

(a) $\sin 390^\circ$  
(b) $\cos 420^\circ$  
(c) $\tan \frac{9\pi}{4}$  
(d) $\sec \left(-\frac{7\pi}{4}\right)$  
(e) $\csc(-270^\circ)$

**Solution**

(a) It is best to sketch the angle first. See Figure 46. The angle $390^\circ$ is coterminal with $30^\circ$.

$$\sin 390^\circ = \sin(30^\circ + 360^\circ) = \sin 30^\circ = \frac{1}{2}$$

(b) See Figure 47. The angle $420^\circ$ is coterminal with $60^\circ$.

$$\cos 420^\circ = \cos(60^\circ + 360^\circ) = \cos 60^\circ = \frac{1}{2}$$

(c) See Figure 48. The angle $\frac{9\pi}{4}$ is coterminal with $\frac{\pi}{4}$.

$$\tan \frac{9\pi}{4} = \tan \left(\frac{\pi}{4} + 2\pi\right) = \tan \frac{\pi}{4} = 1$$

(d) See Figure 49. The angle $-\frac{7\pi}{4}$ is coterminal with $\frac{\pi}{4}$.

$$\sec \left(-\frac{7\pi}{4}\right) = \sec \left(\frac{\pi}{4} + 2\pi(-1)\right) = \sec \frac{\pi}{4} = \sqrt{2}$$

(e) See Figure 50. The angle $-270^\circ$ is coterminal with $90^\circ$.

$$\csc(-270^\circ) = \csc(90^\circ + 360^\circ(-1)) = \csc 90^\circ = 1$$

As Example 3 illustrates, the value of a trigonometric function of any angle is equal to the value of the same trigonometric function of an angle $\theta$ coterminal to it, where $0^\circ \leq \theta < 360^\circ$ (or $0 \leq \theta < 2\pi$). Because the angles $\theta$ and $\theta + 360^\circ k$ (or $\theta + 2\pi k$), where $k$ is any integer, are coterminal, and because the values of the trigonometric functions are equal for coterminal angles, it follows that

<table>
<thead>
<tr>
<th>$\theta$ degrees</th>
<th>$\theta$ radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin(\theta + 360^\circ k)$</td>
<td>$\sin(\theta + 2\pi k)$</td>
</tr>
<tr>
<td>$\cos(\theta + 360^\circ k)$</td>
<td>$\cos(\theta + 2\pi k)$</td>
</tr>
<tr>
<td>$\tan(\theta + 360^\circ k)$</td>
<td>$\tan(\theta + 2\pi k)$</td>
</tr>
<tr>
<td>$\csc(\theta + 360^\circ k)$</td>
<td>$\csc(\theta + 2\pi k)$</td>
</tr>
<tr>
<td>$\sec(\theta + 360^\circ k)$</td>
<td>$\sec(\theta + 2\pi k)$</td>
</tr>
<tr>
<td>$\cot(\theta + 360^\circ k)$</td>
<td>$\cot(\theta + 2\pi k)$</td>
</tr>
</tbody>
</table>

where $k$ is any integer.
3 Determine the Signs of the Trigonometric Functions of an Angle in a Given Quadrant

If \( \theta \) is not a quadrantal angle, then it will lie in a particular quadrant. In such a case, the signs of the \( x \)-coordinate and \( y \)-coordinate of a point on the terminal side of \( \theta \) are known. Because \( r = \sqrt{a^2 + b^2} > 0 \), it follows that the signs of the trigonometric functions of an angle \( \theta \) can be found if we know in which quadrant lies.

For example, if \( \theta \) lies in quadrant II, as shown in Figure 51, then a point \((a, b)\) on the terminal side of \( \theta \) has a negative \( x \)-coordinate and a positive \( y \)-coordinate. Then,

\[
\sin \theta = \frac{b}{r} > 0, \quad \cos \theta = \frac{a}{r} < 0, \quad \tan \theta = \frac{b}{a} < 0 \\
\csc \theta = \frac{r}{b} > 0, \quad \sec \theta = \frac{r}{a} < 0, \quad \cot \theta = \frac{a}{b} < 0
\]

Table 5 lists the signs of the six trigonometric functions for each quadrant. Figure 52 provides two illustrations.

<table>
<thead>
<tr>
<th>Quadrant of ( \theta )</th>
<th>( \sin \theta, \csc \theta )</th>
<th>( \cos \theta, \sec \theta )</th>
<th>( \tan \theta, \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Positive</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>II</td>
<td>Positive</td>
<td>Negative</td>
<td>Negative</td>
</tr>
<tr>
<td>III</td>
<td>Negative</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>IV</td>
<td>Negative</td>
<td>Positive</td>
<td>Negative</td>
</tr>
</tbody>
</table>

**EXAMPLE 4** Finding the Quadrant in Which an Angle Lies

If \( \sin \theta < 0 \) and \( \cos \theta < 0 \), name the quadrant in which the angle \( \theta \) lies.

**Solution**

If \( \sin \theta < 0 \), then \( \theta \) lies in quadrant III or IV. If \( \cos \theta < 0 \), then \( \theta \) lies in quadrant II or III. Therefore, \( \theta \) lies in quadrant III.
Find the Reference Angle of a General Angle

Once we know in which quadrant an angle lies, we know the sign of each trigonometric function of this angle. This information, along with the reference angle, will allow us to evaluate the trigonometric functions of such an angle.

**DEFINITION**

Let \( \theta \) denote an angle that lies in a quadrant. The acute angle formed by the terminal side of \( \theta \) and the \( x \)-axis is called the reference angle for \( \theta \).

Figure 53 illustrates the reference angle for some general angles \( \theta \). Note that a reference angle is always an acute angle. That is, a reference angle has a measure between \( 0^\circ \) and \( 90^\circ \).

Although formulas can be given for calculating reference angles, usually it is easier to find the reference angle for a given angle by making a quick sketch of the angle.

**EXAMPLE 5**

Finding Reference Angles

Find the reference angle for each of the following angles:

(a) \( 150^\circ \)  (b) \( -45^\circ \)  (c) \( \frac{9\pi}{4} \)  (d) \( -\frac{5\pi}{6} \)

**Solution**

(a) Refer to Figure 54. The reference angle for \( 150^\circ \) is \( 30^\circ \).

(b) Refer to Figure 55. The reference angle for \( -45^\circ \) is \( 45^\circ \).

(c) Refer to Figure 56. The reference angle for \( \frac{9\pi}{4} \) is \( \frac{\pi}{4} \).

(d) Refer to Figure 57. The reference angle for \( -\frac{5\pi}{6} \) is \( \frac{\pi}{6} \).

Now Work Problem 41
5 Use a Reference Angle to Find the Exact Value of a Trigonometric Function

The advantage of using reference angles is that, except for the correct sign, the values of the trigonometric functions of a general angle \( \theta \) equal the values of the trigonometric functions of its reference angle.

**Theorem**

*Reference Angles*

If \( \theta \) is an angle that lies in a quadrant and if \( \alpha \) is its reference angle, then

\[
\begin{align*}
\sin \theta &= \pm \sin \alpha & \cos \theta &= \pm \cos \alpha & \tan \theta &= \pm \tan \alpha \\
\csc \theta &= \pm \csc \alpha & \sec \theta &= \pm \sec \alpha & \cot \theta &= \pm \cot \alpha
\end{align*}
\]

where the + or − sign depends on the quadrant in which \( \theta \) lies.

For example, suppose that \( \theta \) lies in quadrant II and \( \alpha \) is its reference angle. See Figure 58. If \((a, b)\) is a point on the terminal side of \( \theta \) and if \( r = \sqrt{a^2 + b^2} \), we have

\[
\begin{align*}
\sin \theta &= \frac{b}{r} = \sin \alpha & \cos \theta &= \frac{a}{r} = -\frac{|a|}{r} = -\cos \alpha
\end{align*}
\]

and so on.

The next example illustrates how the theorem on reference angles is used.

**Example 6** Using the Reference Angle to Find the Exact Value of a Trigonometric Function

Find the exact value of each of the following trigonometric functions using reference angles.

(a) \( \sin 135^\circ \)  
(b) \( \cos 600^\circ \)  
(c) \( \cos \frac{17\pi}{6} \)  
(d) \( \tan \left(-\frac{\pi}{3}\right) \)

**Solution**

(a) Refer to Figure 59. The reference angle for \( 135^\circ \) is \( 45^\circ \) and \( \sin 45^\circ = \frac{\sqrt{2}}{2} \). The angle \( 135^\circ \) is in quadrant II, where the sine function is positive, so

\[
\sin 135^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}
\]

(b) Refer to Figure 60. The reference angle for \( 600^\circ \) is \( 60^\circ \) and \( \cos 60^\circ = \frac{1}{2} \). The angle \( 600^\circ \) is in quadrant III, where the cosine function is negative, so

\[
\cos 600^\circ = -\cos 60^\circ = -\frac{1}{2}
\]

(c) Refer to Figure 61. The reference angle for \( \frac{17\pi}{6} \) is \( \frac{\pi}{6} \) and \( \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \). The angle \( \frac{17\pi}{6} \) is in quadrant II, where the cosine function is negative, so

\[
\cos \frac{17\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}
\]
Finding the Values of the Trigonometric Functions of a General Angle

If the angle $\theta$ is a quadrantal angle, draw the angle, pick a point on its terminal side, and apply the definition of the trigonometric functions.

If the angle $\theta$ lies in a quadrant:

1. Find the reference angle $\alpha$ of $\theta$.
2. Find the value of the trigonometric function at $\alpha$.
3. Adjust the sign (+ or −) according to the sign of the trigonometric function in the quadrant where $\theta$ lies.

Now Work PROBLEMS 59 AND 63

6 Find the Exact Values of Trigonometric Functions of an Angle, Given Information about the Functions

EXAMPLE 7 Finding the Exact Values of Trigonometric Functions

Given that $\cos \theta = -\frac{2}{3}$, $\frac{\pi}{2} < \theta < \pi$, find the exact value of each of the remaining trigonometric functions.

Solution

The angle $\theta$ lies in quadrant II, so we know that $\sin \theta$ and $\csc \theta$ are positive and the other four trigonometric functions are negative. If $\alpha$ is the reference angle for $\theta$, then $\cos \alpha = \frac{2}{3} = \frac{\text{adjacent}}{\text{hypotenuse}}$. The values of the remaining trigonometric functions of the reference angle $\alpha$ can be found by drawing the appropriate triangle and using the Pythagorean Theorem. We use Figure 63 to obtain

\[
\begin{align*}
\sin \alpha &= \frac{\sqrt{5}}{3} \\
\cos \alpha &= \frac{2}{3} \\
\tan \alpha &= \frac{\sqrt{5}}{2} \\
\csc \alpha &= \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} \\
\sec \alpha &= \frac{3}{2} \\
\cot \alpha &= \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}
\end{align*}
\]

Now we assign the appropriate signs to each of these values to find the values of the trigonometric functions of $\theta$.

\[
\begin{align*}
\sin \theta &= \frac{\sqrt{5}}{3} \\
\cos \theta &= -\frac{2}{3} \\
\tan \theta &= -\frac{\sqrt{5}}{2} \\
\csc \theta &= \frac{3\sqrt{5}}{5} \\
\sec \theta &= -\frac{3}{2} \\
\cot \theta &= -\frac{2\sqrt{5}}{5}
\end{align*}
\]

Now Work PROBLEM 89
Finding the Exact Values of Trigonometric Functions

If \( \tan \theta = -4 \) and \( \sin \theta < 0 \), find the exact value of each of the remaining trigonometric functions of \( \theta \).

**Solution**

Since \( \tan \theta = -4 < 0 \) and \( \sin \theta < 0 \), it follows that \( \theta \) lies in quadrant IV. If \( \alpha \) is the reference angle for \( \theta \), then with we find \( r = \sqrt{1^2 + 4^2} = \sqrt{17} \). See Figure 64. Then

\[
\sin \alpha = \frac{4}{\sqrt{17}}, \quad \cos \alpha = \frac{1}{\sqrt{17}}, \quad \tan \alpha = \frac{4}{1} = 4
\]

\[
\csc \alpha = \frac{\sqrt{17}}{4}, \quad \sec \alpha = \frac{\sqrt{17}}{1} = \sqrt{17}, \quad \cot \alpha = \frac{1}{4}
\]

We assign the appropriate sign to each of these to obtain the values of the trigonometric functions of \( \theta \).

\[
\sin \theta = -\frac{4\sqrt{17}}{17}, \quad \cos \theta = \frac{\sqrt{17}}{17}, \quad \tan \theta = -4
\]

\[
\csc \theta = -\frac{\sqrt{17}}{4}, \quad \sec \theta = \sqrt{17}, \quad \cot \theta = -\frac{1}{4}
\]

**EXAMPLE 8**

Finding the Exact Values of Trigonometric Functions

If \( \tan \theta = -4 \) and \( \sin \theta < 0 \), find the exact value of each of the remaining trigonometric functions of \( \theta \).

**Solution**

Since \( \tan \theta = -4 < 0 \) and \( \sin \theta < 0 \), it follows that \( \theta \) lies in quadrant IV. If \( \alpha \) is the reference angle for \( \theta \), then with we find \( r = \sqrt{1^2 + 4^2} = \sqrt{17} \). See Figure 64. Then

\[
\sin \alpha = \frac{4}{\sqrt{17}}, \quad \cos \alpha = \frac{1}{\sqrt{17}}, \quad \tan \alpha = \frac{4}{1} = 4
\]

\[
\csc \alpha = \frac{\sqrt{17}}{4}, \quad \sec \alpha = \frac{\sqrt{17}}{1} = \sqrt{17}, \quad \cot \alpha = \frac{1}{4}
\]

We assign the appropriate sign to each of these to obtain the values of the trigonometric functions of \( \theta \).

\[
\sin \theta = -\frac{4\sqrt{17}}{17}, \quad \cos \theta = \frac{\sqrt{17}}{17}, \quad \tan \theta = -4
\]

\[
\csc \theta = -\frac{\sqrt{17}}{4}, \quad \sec \theta = \sqrt{17}, \quad \cot \theta = -\frac{1}{4}
\]

Now Work Problem 99

### 7.4 Assess Your Understanding

#### Concepts and Vocabulary

1. For an angle \( \theta \) that lies in quadrant III, the trigonometric functions _____ and _____ are positive.
2. Two angles in standard position that have the same terminal side are _____.
3. The reference angle of 240° is _____.
4. True or False \( \sin 182° = \cos 2° \).
5. True or False The reference angle is always an acute angle.
6. True or False The reference angle is always an acute angle.
7. What is the reference angle of 600°?
8. In which quadrants is the cosine function positive?
9. If \( \theta \) lies in quadrant II, what is an expression for \( \tan \theta \) in terms of \( \sin \theta \) and \( \cos \theta \)?
10. What is the reference angle of \( \frac{13\pi}{3} \)?

#### Skill Building

In Problems 11–20, a point on the terminal side of an angle \( \theta \) is given. Find the exact value of each of the six trigonometric functions of \( \theta \).

11. \( (3, 4) \)
12. \( (5, -12) \)
13. \( (2, -3) \)
14. \( (-1, -2) \)
15. \( (-3, -3) \)
16. \( (2, -2) \)
17. \( \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \)
18. \( \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \)
19. \( \left( \sqrt{2}, -\sqrt{2} \right) \)
20. \( \left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \)

In Problems 21–32, use a coterminal angle to find the exact value of each expression. Do not use a calculator.

21. \( \cos 405° \)
22. \( \tan 405° \)
23. \( \sin 405° \)
24. \( \sin 390° \)
25. \( \csc 450° \)
26. \( \sec 540° \)
27. \( \cot 390° \)
28. \( \sec 420° \)
29. \( \cos \frac{3\pi}{4} \)
30. \( \sin \frac{9\pi}{4} \)
31. \( \tan(2\pi) \)
32. \( \csc \frac{9\pi}{2} \)

In Problems 33–40, name the quadrant in which the angle \( \theta \) lies.

33. \( \sin \theta > 0, \cos \theta < 0 \)
34. \( \sin \theta < 0, \cos \theta > 0 \)
35. \( \sin \theta < 0, \tan \theta < 0 \)
36. \( \cos \theta > 0, \tan \theta > 0 \)
37. \( \cos \theta > 0, \cot \theta < 0 \)
38. \( \sin \theta < 0, \cot \theta > 0 \)
39. \( \sec \theta < 0, \tan \theta > 0 \)
40. \( \csc \theta > 0, \cot \theta < 0 \)
In Problems 41–58, find the reference angle of each angle.

101. \( \tan 40° \)

102. \( \sin 120° \)

103. \( \cos 300° \)

104. \( \sin 210° \)

105. \( \tan 140° \)

106. \( \cot 5π \)

In Problems 59–88, use the reference angle to find the exact value of each expression. Do not use a calculator.

59. \( \sin 150° \)

60. \( \cos 210° \)

61. \( \cos 315° \)

62. \( \sin 120° \)

63. \( \sin 180° \)

64. \( \cos 600° \)

65. \( \cos (-45°) \)

66. \( \sin (-240°) \)

67. \( \sec 240° \)

68. \( \csc 300° \)

69. \( \cot 330° \)

70. \( \tan 225° \)

71. \( \sin \left( \frac{3π}{4} \right) \)

72. \( \cos \left( \frac{2π}{3} \right) \)

73. \( \cot \left( \frac{7π}{6} \right) \)

74. \( \csc \left( \frac{7π}{4} \right) \)

75. \( \sec \left( \frac{13π}{4} \right) \)

76. \( \tan \left( \frac{8π}{3} \right) \)

77. \( \sin \left( \frac{2π}{3} \right) \)

78. \( \cot \left( \frac{5π}{6} \right) \)

79. \( \tan \left( \frac{14π}{3} \right) \)

80. \( \sec \left( \frac{11π}{4} \right) \)

81. \( \csc \left( \frac{7π}{4} \right) \)

82. \( \sec \left( \frac{5π}{2} \right) \)

83. \( \sin (8π) \)

84. \( \cos (-2π) \)

85. \( \tan(7π) \)

86. \( \cot(5π) \)

87. \( \sec(-3π) \)

88. \( \csc \left( \frac{5π}{2} \right) \)

In Problems 89–106, find the exact value of each of the remaining trigonometric functions of \( \theta \).

89. \( \sin \theta = \frac{12}{13}, \) \( \theta \) in Quadrant II

90. \( \cos \theta = \frac{3}{5}, \) \( \theta \) in Quadrant IV

91. \( \cos \theta = \frac{4}{5}, \) \( \theta \) in Quadrant III

92. \( \sin \theta = -\frac{5}{13}, \) \( \theta \) in Quadrant III

93. \( \sin \theta = \frac{5}{13}, \) \( 0° < \theta < 180° \)

94. \( \cos \theta = \frac{4}{5}, \) \( 270° < \theta < 360° \)

95. \( \cos \theta = -\frac{1}{3}, \) \( 180° < \theta < 360° \)

96. \( \sin \theta = \frac{2}{3}, \) \( 180° < \theta < 270° \)

97. \( \sin \theta = \frac{2}{3}, \) \( \tan \theta < 0 \)

98. \( \cos \theta = -\frac{1}{4}, \) \( \tan \theta > 0 \)

99. \( \sec \theta = 2, \) \( \sin \theta > 0 \)

100. \( \sec \theta = 3, \) \( \cot \theta < 0 \)

101. \( \tan \theta = \frac{3}{4}, \) \( \sin \theta < 0 \)

102. \( \cot \theta = \frac{4}{3}, \) \( \cos \theta > 0 \)

103. \( \tan \theta = -\frac{1}{3}, \) \( \sin \theta > 0 \)

104. \( \sec \theta = -2, \) \( \tan \theta > 0 \)

105. \( \csc \theta = -2, \) \( \tan \theta < 0 \)

106. \( \cot \theta = -2, \) \( \sec \theta > 0 \)

107. Find the exact value of \( \sin 40° + \sin 130° + \sin 220° + \sin 310° \).

108. Find the exact value of \( \tan 40° + \tan 140° \).

Applications and Extensions

109. If \( f(\theta) = \sin \theta = 0.2 \), find \( f(\theta + π) \).

110. If \( g(\theta) = \cos \theta = 0.4 \), find \( g(\theta + π) \).

111. If \( F(\theta) = \tan \theta = 3 \), find \( F(\theta + π) \).

112. If \( G(\theta) = \cot \theta = -2 \), find \( G(\theta + π) \).

113. If \( \sin \theta = \frac{1}{5} \), find \( \csc (\theta + π) \).

114. If \( \cos \theta = \frac{2}{3} \), find \( \sec (\theta + π) \).

115. Find the exact value of \( \sin 1° + \sin 2° + \sin 3° + \cdots + \sin 358° + \sin 359° \).

116. Find the exact value of \( \cos 1° + \cos 2° + \cos 3° + \cdots + \cos 358° + \cos 359° \).

117. Projectile Motion An object is propelled upward at an angle \( \theta, 45° < \theta < 90° \), to the horizontal with an initial velocity of \( v_0 \) feet per second from the base of a plane that makes an angle of 45° with the horizontal. See the illustration. If air resistance is ignored, the distance \( R \) that it travels up the inclined plane is given by the function

\[
R(\theta) = \frac{v_0^2 \sqrt{2}}{32} \left[ \sin(2\theta) - \cos(2\theta) - 1 \right]
\]

(a) Find the distance \( R \) that the object travels along the inclined plane if the initial velocity is 32 feet per second and \( \theta = 60° \).

(b) Graph \( R = R(\theta) \) if the initial velocity is 32 feet per second.

(c) What value of \( \theta \) makes \( R \) largest?

Discussion and Writing

118. Give three examples that demonstrate how to use the theorem on reference angles.

119. Write a brief paragraph that explains how to quickly compute the value of the trigonometric functions of 0°, 90°, 180°, and 270°.
In this section, we develop important properties of the trigonometric functions. We begin by introducing the trigonometric functions using the unit circle. This approach will lead to the definition given earlier of the trigonometric functions of a general angle.

1. **Find the Exact Values of the Trigonometric Functions Using the Unit Circle**

Recall that the unit circle is a circle whose radius is 1 and whose center is at the origin of a rectangular coordinate system. Also recall that any circle of radius \( r \) has circumference of length \( 2\pi r \). Therefore, the unit circle (radius = 1) has a circumference of length \( 2\pi \). In other words, for 1 revolution around the unit circle the length of the arc is \( 2\pi \) units.

The following discussion sets the stage for defining the trigonometric functions using the unit circle.

Let \( t \geq 0 \) be any real number and let \( s \) be the distance from the origin to \( t \) on the real number line. See the red portion of Figure 65(a). Now look at the unit circle in Figure 65(a). Beginning at the point \((1, 0)\) on the unit circle, travel \( s = t \) units in the counterclockwise direction along the circle to arrive at the point \( P = (a, b) \). In this sense, the length \( s = t \) units is being wrapped around the unit circle.

If \( t < 0 \), we begin at the point \((1, 0)\) on the unit circle and travel \( s = |t| \) units in the clockwise direction to arrive at the point \( P = (a, b) \). See Figure 65(b).

If \( t > 2\pi \) or if \( t < -2\pi \), it will be necessary to travel around the unit circle more than once before arriving at point \( P \). Do you see why?
Let’s describe this process another way. Picture a string of length \( s = |t| \) units being wrapped around a circle of radius 1 unit. We start wrapping the string around the circle at the point \((1, 0)\). If \( t > 0 \), we wrap the string in the counterclockwise direction; if \( t < 0 \), we wrap the string in the clockwise direction. The point \( P = (a, b) \) is the point where the string ends.

This discussion tells us that, for any real number \( t \), we can locate a unique point on the unit circle. We call this point the point \( P \) on the unit circle that corresponds to \( t \). This is the important idea here. No matter what real number \( t \) is chosen, there is a unique point \( P \) on the unit circle corresponding to it. We use the coordinates of the point \( P = (a, b) \) on the unit circle corresponding to the real number \( t \) to define the six trigonometric functions of \( t \).

Let \( t \) be a real number and let \( P = (a, b) \) be the point on the unit circle that corresponds to \( t \).

The sine function associates with \( t \) the \( y \)-coordinate of \( P \) and is denoted by

\[
\sin t = b
\]

The cosine function associates with \( t \) the \( x \)-coordinate of \( P \) and is denoted by

\[
\cos t = a
\]

If \( a \neq 0 \), the tangent function is defined as

\[
\tan t = \frac{b}{a}
\]

If \( b \neq 0 \), the cosecant function is defined as

\[
\csc t = \frac{1}{b}
\]

If \( a \neq 0 \), the secant function is defined as

\[
\sec t = \frac{1}{a}
\]

If \( b \neq 0 \), the cotangent function is defined as

\[
\cot t = \frac{a}{b}
\]

Once again, notice in these definitions that if \( a = 0 \) (that is, if the point \( P \) is on the \( y \)-axis) the tangent function and the secant function are undefined. Also, if \( b = 0 \) (that is, if the point \( P \) is on the \( x \)-axis), the cosecant function and the cotangent function are undefined.

Because we use the unit circle in these definitions of the trigonometric functions, they are also sometimes referred to as circular functions.

**EXAMPLE 1**

**Finding the Values of the Trigonometric Functions Using a Point on the Unit Circle**

Find the values of \( \sin t \), \( \cos t \), \( \tan t \), \( \csc t \), \( \sec t \), and \( \cot t \) if \( P = \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \) is the point on the unit circle that corresponds to the real number \( t \).
Solution

See Figure 66. We follow the definition of the six trigonometric functions using \( P = \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right) = (a, b) \). Then, with \( a = -\frac{1}{2} \) and \( b = \frac{\sqrt{3}}{2} \), we have

\[
\begin{align*}
\sin t &= b = \frac{\sqrt{3}}{2} \\
\cos t &= a = -\frac{1}{2} \\
\tan t &= \frac{b}{a} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3} \\
\csc t &= \frac{1}{b} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\
\sec t &= \frac{1}{a} = \frac{1}{-\frac{1}{2}} = -2 \\
\cot t &= \frac{a}{b} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}
\end{align*}
\]

Now Work Problem 9

Trigonometric Functions of Angles

Let \( P = (a, b) \) be the point on the unit circle corresponding to the real number \( t \). See Figure 67(a). Let \( \theta \) be the angle in standard position, measured in radians, whose terminal side is the ray from the origin through \( P \). See Figure 67(b). Since the unit circle has radius 1 unit, if \( s = |t| \) units, then from the arc length formula \( s = r\theta \), we have \( \theta = t \) radians. See Figures 67(c) and (d).

The point \( P = (a, b) \) on the unit circle that corresponds to the real number \( t \) is the point \( P \) on the terminal side of the angle \( \theta = t \) radians. As a result, we can say that

\[
\sin t = \sin \theta \\
\cos t = \cos \theta \\
\tan t = \tan \theta \\
\csc t = \csc \theta \\
\sec t = \sec \theta \\
\cot t = \cot \theta
\]

and so on. We can now define the trigonometric functions of the angle \( \theta \).

**DEFINITION**

If \( \theta = t \) radians, the six **trigonometric functions of the angle \( \theta \)** are defined as

\[
\begin{align*}
\sin \theta &= \sin t \\
\cos \theta &= \cos t \\
\tan \theta &= \tan t \\
\csc \theta &= \csc t \\
\sec \theta &= \sec t \\
\cot \theta &= \cot t
\end{align*}
\]

Even though the trigonometric functions can be viewed both as functions of real numbers and as functions of angles, it is customary to refer to trigonometric functions of real numbers and trigonometric functions of angles collectively as the **trigonometric functions**. We will follow this practice from now on.

Since the values of the trigonometric functions of an angle \( \theta \) are determined by the coordinates of the point \( P = (a, b) \) on the unit circle corresponding to \( \theta \), the units used to measure the angle \( \theta \) are irrelevant. For example, it does not matter...
whether we write \( \theta = \frac{\pi}{2} \) radians or \( \theta = 90^\circ \). In either case, the point on the unit circle corresponding to this angle is \( P = (0, 1) \). As a result,

\[
\sin \frac{\pi}{2} = \sin 90^\circ = 1 \quad \text{and} \quad \cos \frac{\pi}{2} = \cos 90^\circ = 0
\]

To find the exact value of a trigonometric function of an angle \( \theta \) requires that we locate the corresponding point \( P^* = (a^*, b^*) \) on the unit circle. In fact, though, any circle whose center is at the origin can be used.

Let \( \theta \) be any nonquadrantal angle placed in standard position. Let \( P = (a, b) \) be the point on the circle \( x^2 + y^2 = r^2 \) that corresponds to \( \theta \) and let \( P^* = (a^*, b^*) \) be the point on the unit circle that corresponds to \( \theta \). See Figure 68.

Notice that the triangles \( OAP^* \) and \( OAP \) are similar so ratios of corresponding sides are equal.

\[
\begin{align*}
\frac{b^*}{r} &= \frac{b}{r} \quad \frac{a^*}{r} = \frac{a}{r} \quad \frac{b^*}{a^*} = \frac{b}{a} \\
\frac{1}{r^*} &= \frac{1}{r} \quad \frac{1}{a^*} = \frac{1}{a} \quad \frac{b^*}{a^*} = \frac{b}{a}
\end{align*}
\]

These results lead us to formulate the following theorem:

**Theorem**

For an angle \( \theta \) in standard position, let \( P = (a, b) \) be any point on the terminal side of \( \theta \) that is also on the circle \( x^2 + y^2 = r^2 \). Then

\[
\begin{align*}
\sin \theta &= \frac{b}{r} \\
\cos \theta &= \frac{a}{r} \\
\tan \theta &= \frac{b}{a} \quad a \neq 0 \\
csc \theta &= \frac{r}{b} \quad b \neq 0 \\
\sec \theta &= \frac{r}{a} \quad a \neq 0 \\
\cot \theta &= \frac{a}{b} \quad b \neq 0
\end{align*}
\]

This result coincides with the definition given in Section 7.4 for the six trigonometric functions of a general angle \( \theta \).

**Now Work Problem 15**

2. **Know the Domain and Range of the Trigonometric Functions**

Let \( \theta \) be an angle in standard position, and let \( P = (a, b) \) be the point on the unit circle that corresponds to \( \theta \). See Figure 69. Then, by the definition given earlier:

\[
\begin{align*}
\sin \theta &= b \\
\cos \theta &= a \\
\tan \theta &= \frac{b}{a} \quad a \neq 0 \\
csc \theta &= \frac{1}{b} \quad b \neq 0 \\
\sec \theta &= \frac{1}{a} \quad a \neq 0 \\
\cot \theta &= \frac{a}{b} \quad b \neq 0
\end{align*}
\]

For \( \sin \theta \) and \( \cos \theta \), \( \theta \) can be any angle, so it follows that the domain of the sine function and cosine function is the set of all real numbers.

The domain of the sine function is the set of all real numbers.
The domain of the cosine function is the set of all real numbers.

If \( a = 0 \), then the tangent function and the secant function are not defined. That is, for the tangent function and secant function, the \( x \)-coordinate of \( P = (a, b) \) cannot be 0. On the unit circle, there are two such points, \((0, 1)\) and \((0, -1)\). These two points correspond to the angles \( \frac{\pi}{2} (90^\circ) \) and \( \frac{3\pi}{2} (270^\circ) \) or, more generally, to any angle that is an odd integer multiple of \( \frac{\pi}{2} (90^\circ) \), such as \( \pm \frac{\pi}{2} (\pm 90^\circ) \), \( \pm \frac{3\pi}{2} (\pm 270^\circ) \), \( \pm \frac{5\pi}{2} (\pm 450^\circ) \), \( \pm \frac{7\pi}{2} (\pm 630^\circ) \), \( \pm \frac{9\pi}{2} (\pm 810^\circ) \), and so forth.
and \( \pm \frac{5\pi}{2} (\pm 450^\circ) \). Such angles must therefore be excluded from the domain of the tangent function and secant function.

The domain of the tangent function is the set of all real numbers, except odd integer multiples of \( \frac{\pi}{2} (90^\circ) \).

The domain of the secant function is the set of all real numbers, except odd integer multiples of \( \frac{\pi}{2} (90^\circ) \).

If \( b = 0 \), then the cotangent function and the cosecant function are not defined. For the cotangent function and cosecant function, the \( y \)-coordinate of \( P = (a, b) \) cannot be 0. On the unit circle, there are two such points, \((1, 0)\) and \((-1, 0)\). These two points correspond to the angles \( 0 (0^\circ) \) and \( \pi (180^\circ) \) or, more generally, to any angle that is an integer multiple of \( \pi (180^\circ) \), such as \( 0 (0^\circ) \), \( \pm \pi (\pm 180^\circ) \), \( \pm 2\pi (\pm 360^\circ) \), and \( \pm 3\pi (\pm 540^\circ) \). Such angles must be excluded from the domain of the cotangent function and cosecant function.

The domain of the cotangent function is the set of all real numbers, except integer multiples of \( \pi (180^\circ) \).

The domain of the cosecant function is the set of all real numbers, except integer multiples of \( \pi (180^\circ) \).

Next, we determine the range of each of the six trigonometric functions. Refer again to Figure 69. Let \( P = (a, b) \) be the point on the unit circle that corresponds to the angle \( \theta \). It follows that \( -1 \leq a \leq 1 \) and \( -1 \leq b \leq 1 \). Since \( \sin \theta = b \) and \( \cos \theta = a \), we have

\[-1 \leq \sin \theta \leq 1 \quad \text{and} \quad -1 \leq \cos \theta \leq 1\]

The range of both the sine function and the cosine function consists of all real numbers between \(-1\) and 1, inclusive. Using absolute value notation, we have \( |\sin \theta| \leq 1 \) and \( |\cos \theta| \leq 1 \).

If \( \theta \) is not an integer multiple of \( \pi (180^\circ) \), then \( \csc \theta = \frac{1}{b} \). Since \( b = \sin \theta \) and \( |b| = |\sin \theta| \leq 1 \), it follows that \( |\csc \theta| = \frac{1}{|\sin \theta|} = \frac{1}{|b|} \geq 1 \). The range of the cosecant function consists of all real numbers less than or equal to \(-1\) or greater than or equal to \(1\). That is,

\[\csc \theta \leq -1 \quad \text{or} \quad \csc \theta \geq 1\]

If \( \theta \) is not an odd integer multiple of \( \frac{\pi}{2} (90^\circ) \), then \( \sec \theta = \frac{1}{a} \). Since \( a = \cos \theta \) and \( |a| = |\cos \theta| \leq 1 \), it follows that \( |\sec \theta| = \frac{1}{|\cos \theta|} = \frac{1}{|a|} \leq 1 \). The range of the secant function consists of all real numbers less than or equal to \(-1\) or greater than or equal to \(1\). That is,

\[\sec \theta \leq -1 \quad \text{or} \quad \sec \theta \geq 1\]

The range of both the tangent function and the cotangent function consists of all real numbers. That is,

\[-\infty < \tan \theta < \infty \quad \text{and} \quad -\infty < \cot \theta < \infty\]

You are asked to prove this in Problems 93 and 94.

Table 6 summarizes these results.
### Table 6

<table>
<thead>
<tr>
<th>Function</th>
<th>Symbol</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>sine</td>
<td>( f(\theta) = \sin \theta )</td>
<td>All real numbers</td>
<td>All real numbers from (-1) to (1), inclusive</td>
</tr>
<tr>
<td>cosine</td>
<td>( f(\theta) = \cos \theta )</td>
<td>All real numbers</td>
<td>All real numbers from (-1) to (1), inclusive</td>
</tr>
<tr>
<td>tangent</td>
<td>( f(\theta) = \tan \theta )</td>
<td>All real numbers, except odd integer multiples of (\frac{\pi}{2})((90^\circ))</td>
<td>All real numbers</td>
</tr>
<tr>
<td>cosecant</td>
<td>( f(\theta) = \csc \theta )</td>
<td>All real numbers, except integer multiples of (\pi)((180^\circ))</td>
<td>All real numbers greater than or equal to (1) or less than or equal to (-1)</td>
</tr>
<tr>
<td>secant</td>
<td>( f(\theta) = \sec \theta )</td>
<td>All real numbers, except odd integer multiples of (\frac{\pi}{2})((90^\circ))</td>
<td>All real numbers greater than or equal to (1) or less than or equal to (-1)</td>
</tr>
<tr>
<td>cotangent</td>
<td>( f(\theta) = \cot \theta )</td>
<td>All real numbers, except integer multiples of (\pi)((180^\circ))</td>
<td>All real numbers</td>
</tr>
</tbody>
</table>

### Now Work

**Problems 61 and 65**

3 **Use the Periodic Properties to Find the Exact Values of the Trigonometric Functions**

Look at Figure 70. This figure shows that for an angle of \(\frac{\pi}{3}\) radians the corresponding point \(P\) on the unit circle is \((\frac{1}{2}, \frac{\sqrt{3}}{2})\). Notice that for an angle of \(\frac{\pi}{3} + 2\pi\) radians, the corresponding point \(P\) on the unit circle is also \((\frac{1}{2}, \frac{\sqrt{3}}{2})\). As a result,

\[
\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin \left(\frac{\pi}{3} + 2\pi\right) = \frac{\sqrt{3}}{2} \\
\cos \frac{\pi}{3} = \frac{1}{2} \quad \text{and} \quad \cos \left(\frac{\pi}{3} + 2\pi\right) = \frac{1}{2}
\]

This example illustrates a more general situation. For a given angle \(\theta\), measured in radians, suppose that we know the corresponding point \(P = (a, b)\) on the unit circle. Now add \(2\pi\) to \(\theta\). The point on the unit circle corresponding to \(\theta + 2\pi\) is identical to the point \(P\) on the unit circle corresponding to \(\theta\). See Figure 71. The values of the trigonometric functions of \(\theta + 2\pi\) are equal to the values of the corresponding trigonometric functions of \(\theta\).

If we add (or subtract) integer multiples of \(2\pi\) to \(\theta\), the values of the sine and cosine function remain unchanged. That is, for all \(\theta\)

\[
\sin(\theta + 2\pi k) = \sin \theta \quad \cos(\theta + 2\pi k) = \cos \theta \\
\text{where } k \text{ is any integer} \quad (1)
\]

Functions that exhibit this kind of behavior are called **periodic functions**.

**Definition**

A function \(f\) is called periodic if there is a positive number \(p\) such that, whenever \(\theta\) is in the domain of \(f\), so is \(\theta + p\), and

\[
f(\theta + p) = f(\theta)
\]

If there is a smallest such number \(p\), this smallest value is called the **(fundamental) period** of \(f\).

Based on equation (1), the sine and cosine functions are periodic. In fact, the sine, cosine, secant, and cosecant functions have period \(2\pi\). You are asked to prove this in Problems 93 through 96.
The tangent and cotangent functions are periodic with period $\pi$. See Figure 72 for a partial justification. You are asked to prove this statement in Problems 97 and 98.

These facts are summarized as follows:

**Periodic Properties**

\[
\begin{align*}
\sin(\theta + 2\pi) &= \sin \theta & \cos(\theta + 2\pi) &= \cos \theta & \tan(\theta + 2\pi) &= \tan \theta \\
\csc(\theta + 2\pi) &= \csc \theta & \sec(\theta + 2\pi) &= \sec \theta & \cot(\theta + 2\pi) &= \cot \theta \\
\end{align*}
\]

Because the sine, cosine, secant, and cosecant functions have period $2\pi$, once we know their values for $0 \leq \theta < 2\pi$, we know all their values; similarly, since the tangent and cotangent functions have period $\pi$, once we know their values for $0 \leq \theta < \pi$, we know all their values.

**Example 2**

**Using Periodic Properties to Find Exact Values**

Find the exact value of:

(a) $\sin 420^\circ$  
(b) $\tan \frac{5\pi}{4}$  
(c) $\cos \frac{11\pi}{4}$

**Solution**

(a) $\sin 420^\circ = \sin (60^\circ + 360^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$

(b) $\tan \frac{5\pi}{4} = \tan \left( \frac{\pi}{4} + \pi \right) = \tan \frac{\pi}{4} = 1$

(c) $\cos \frac{11\pi}{4} = \cos \left( \frac{3\pi}{4} + \frac{8\pi}{4} \right) = \cos \left( \frac{3\pi}{4} + 2\pi \right) = \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

The periodic properties of the trigonometric functions will be very helpful to us when we study their graphs in the next section.

**Now Work** PROBLEMS 21 AND 79

4. **Use Even–Odd Properties to Find the Exact Values of the Trigonometric Functions**

Recall that a function $f$ is even if $f(-\theta) = f(\theta)$ for all $\theta$ in the domain of $f$; a function $f$ is odd if $f(-\theta) = -f(\theta)$ for all $\theta$ in the domain of $f$. We will now show that the trigonometric functions sine, tangent, cotangent, and cosecant are odd functions and the functions cosine and secant are even functions.

**Theorem**

**Even–Odd Properties**

\[
\begin{align*}
\sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta & \tan(-\theta) &= -\tan \theta \\
\csc(-\theta) &= -\csc \theta & \sec(-\theta) &= \sec \theta & \cot(-\theta) &= -\cot \theta \\
\end{align*}
\]

**Proof**

Let $P = (a, b)$ be the point on the unit circle that corresponds to the angle $\theta$. See Figure 73. The point $Q$ on the unit circle that corresponds to the angle $-\theta$ will have coordinates $(a, -b)$. Using the definition for the trigonometric functions, we have

\[
\begin{align*}
\sin \theta &= b & \sin(-\theta) &= -b & \cos \theta &= a & \cos(-\theta) &= a \\
\end{align*}
\]
so

\[
\sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta
\]

Now, using these results and some of the Fundamental Identities, we have

\[
\begin{align*}
\tan(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin \theta}{\cos \theta} = -\tan \theta \\
\cot(-\theta) &= \frac{1}{\tan(-\theta)} = \frac{1}{-\tan \theta} = -\cot \theta \\
\sec(-\theta) &= \frac{1}{\cos(-\theta)} = \frac{1}{\cos \theta} = \sec \theta \\
\csc(-\theta) &= \frac{1}{\sin(-\theta)} = \frac{1}{-\sin \theta} = -\csc \theta
\end{align*}
\]

**EXAMPLE 3**

**Finding Exact Values Using Even–Odd Properties**

Find the exact value of:

(a) \(\sin(-45^\circ)\) \quad (b) \(\cos(-\pi)\) \quad (c) \(\cot\left(-\frac{3\pi}{2}\right)\) \quad (d) \(\tan\left(-\frac{37\pi}{4}\right)\)

**Solution**

(a) \(\sin(-45^\circ) = -\sin 45^\circ = -\frac{\sqrt{2}}{2}\)

Odd function

(b) \(\cos(-\pi) = \cos \pi = -1\)

Even function

(c) \(\cot\left(-\frac{3\pi}{2}\right) = -\cot\frac{3\pi}{2} = 0\)

Odd function

(d) \(\tan\left(-\frac{37\pi}{4}\right) = -\tan\frac{37\pi}{4} = -\tan\left(\frac{\pi}{4} + 9\pi\right) = -\tan\frac{\pi}{4} = -1\)

Odd function

Period is \(\pi\).

**Now Work** 

**PROBLEMS 37 AND 73**

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**7.5 Assess Your Understanding**

**‘Are You Prepared?’** *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. What is the equation of the unit circle? (p. 190)
2. The domain of the function \(f(x) = \frac{3x - 6}{x - 4}\) is _______.
3. A function for which \(f(x) = f(-x)\) for all \(x\) in the domain of \(f\) is called a(n) _______ function. (pp. 231–233)

**Concepts and Vocabulary**

4. The sine, cosine, cosecant, and secant functions have period _______; the tangent and cotangent functions have period _______.
5. The domain of the tangent function is _______.
6. The range of the sine function is _______.
7. If \(\sin \theta = 0.2\), then \(\sin(-\theta) = \) _______ and \(\sin(\theta + 2\pi) = \) _______.
8. **True or False** The only even trigonometric functions are the cosine and secant functions.
Skills Building

In Problems 9–14, the point P on the unit circle that corresponds to a real number t is given. Find sin t, cos t, tan t, csc t, sec t, and cot t.

9. \(\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\)
10. \(\left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right)\)
11. \(\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\)
12. \(\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\)
13. \(\left(\frac{5}{3}, -\frac{2}{3}\right)\)
14. \(\left(-\frac{5}{3}, \frac{2\sqrt{5}}{5}\right)\)

In Problems 15–20, the point P on the circle \(x^2 + y^2 = r^2\) that is also on the terminal side of an angle \(\theta\) in standard position is given. Find sin \(\theta\), cos \(\theta\), tan \(\theta\), csc \(\theta\), sec \(\theta\), and cot \(\theta\).

15. \((3, -4)\)
16. \((-4, -3)\)
17. \((-2, 3)\)
18. \((2, -4)\)
19. \((-1, -1)\)
20. \((-3, 1)\)

In Problems 21–36, use the fact that the trigonometric functions are periodic to find the exact value of each expression. Do not use a calculator.

21. \(\sin 405^\circ\)
22. \(\cos 405^\circ\)
23. \(\tan 405^\circ\)
24. \(\sin 390^\circ\)
25. \(\csc 450^\circ\)
26. \(\sec 540^\circ\)
27. \(\cot 390^\circ\)
28. \(\sec 420^\circ\)
29. \(\cos \frac{3\pi}{4}\)
30. \(\sin \frac{9\pi}{4}\)
31. \(\tan(21\pi)\)
32. \(\csc \frac{9\pi}{2}\)
33. \(\sec \frac{17\pi}{4}\)
34. \(\cot \frac{17\pi}{4}\)
35. \(\tan \frac{19\pi}{6}\)
36. \(\sec \frac{25\pi}{6}\)

In Problems 37–54, use the even–odd properties to find the exact value of each expression. Do not use a calculator.

37. \(\sin(-60^\circ)\)
38. \(\cos(-30^\circ)\)
39. \(\tan(-30^\circ)\)
40. \(\sin(-135^\circ)\)
41. \(\sec(-60^\circ)\)
42. \(\csc(-30^\circ)\)
43. \(\sin(-90^\circ)\)
44. \(\cos(-270^\circ)\)
45. \(\tan\left(-\frac{\pi}{4}\right)\)
46. \(\sin(-\pi)\)
47. \(\cos\left(-\frac{\pi}{4}\right)\)
48. \(\sin\left(-\frac{\pi}{3}\right)\)
49. \(\tan(-\pi)\)
50. \(\sin\left(-\frac{3\pi}{2}\right)\)
51. \(\csc\left(-\frac{\pi}{4}\right)\)
52. \(\sec(-\pi)\)
53. \(\sec\left(-\frac{\pi}{6}\right)\)
54. \(\csc\left(-\frac{\pi}{3}\right)\)

In Problems 55–60, find the exact value of each expression. Do not use a calculator.

55. \(\sin(-\pi) + \cos(5\pi)\)
56. \(\tan\left(-\frac{5\pi}{6}\right) - \cot\left(-\frac{7\pi}{2}\right)\)
57. \(\sec(-\pi) + \cos\left(-\frac{\pi}{2}\right)\)
58. \(\tan(-6\pi) + \cos\left(-\frac{9\pi}{4}\right)\)
59. \(\sin\left(-\frac{9\pi}{4}\right) - \tan\left(-\frac{9\pi}{4}\right)\)
60. \(\cos\left(-\frac{17\pi}{4}\right) - \sin\left(-\frac{3\pi}{2}\right)\)

61. What is the domain of the sine function?
62. What is the domain of the cosine function?
63. For what numbers \(\theta\) is \(f(\theta) = \tan \theta\) not defined?
64. For what numbers \(\theta\) is \(f(\theta) = \cot \theta\) not defined?
65. For what numbers \(\theta\) is \(f(\theta) = \sec \theta\) not defined?
66. For what numbers \(\theta\) is \(f(\theta) = \csc \theta\) not defined?
67. What is the range of the sine function?
68. What is the range of the cosine function?
69. What is the range of the tangent function?
70. What is the range of the cotangent function?
71. What is the range of the secant function?
72. What is the range of the cosecant function?
73. Is the sine function even, odd, or neither? Is its graph symmetric? With respect to what?
74. Is the cosine function even, odd, or neither? Is its graph symmetric? With respect to what?
75. Is the tangent function even, odd, or neither? Is its graph symmetric? With respect to what?
76. Is the cotangent function even, odd, or neither? Is its graph symmetric? With respect to what?
77. Is the secant function even, odd, or neither? Is its graph symmetric? With respect to what?
78. Is the cosecant function even, odd, or neither? Is its graph symmetric? With respect to what?
79. If \(\sin \theta = 0.3\), find the value of:
\[\sin \theta + \sin(\theta + 2\pi) + \sin(\theta + 4\pi)\]
80. If \(\cos \theta = 0.2\), find the value of:
\[\cos \theta + \cos(\theta + 2\pi) + \cos(\theta + 4\pi)\]
81. If \( \tan \theta = 3 \), find the value of:
\[
\tan \theta + \tan(\theta + \pi) + \tan(\theta + 2\pi)
\]
82. If \( \cot \theta = -2 \), find the value of:
\[
\cot \theta + \cot(\theta - \pi) + \cot(\theta - 2\pi)
\]

In Problems 83–88, use the periodic and even–odd properties.
83. If \( f(x) = \sin x \) and \( f(a) = \frac{1}{3} \), find the exact value of:
(a) \( f(-a) \)  
(b) \( f(a) + f(a + 2\pi) + f(a + 4\pi) \)
84. If \( f(x) = \cos x \) and \( f(a) = \frac{1}{4} \), find the exact value of:
(a) \( f(-a) \)  
(b) \( f(a) + f(a + 2\pi) + f(a - 2\pi) \)
85. If \( f(x) = \tan x \) and \( f(a) = 2 \), find the exact value of:
(a) \( f(-a) \)  
(b) \( f(a) + f(a + \pi) + f(a + 2\pi) \)
86. If \( f(x) = \cot x \) and \( f(a) = -3 \), find the exact value of:
(a) \( f(-a) \)  
(b) \( f(a) + f(a + \pi) + f(a + 4\pi) \)
87. If \( f(x) = \sec x \) and \( f(a) = -4 \), find the exact value of:
(a) \( f(-a) \)  
(b) \( f(a) + f(a + 2\pi) + f(a + 4\pi) \)
88. If \( f(x) = \csc x \) and \( f(a) = 2 \), find the exact value of:
(a) \( f(-a) \)  
(b) \( f(a) + f(a + 2\pi) + f(a + 4\pi) \)

Applications and Extensions
In Problems 89–92, use the figure to approximate the value of the six trigonometric functions at \( t \) to the nearest tenth. Then use a calculator to approximate each of the six trigonometric functions at \( t \).

![Unit Circle](image)

89. (a) \( t = 1 \)  
(b) \( t = 5.1 \)
90. (a) \( t = 2 \)  
(b) \( t = 4 \)
91. Show that the range of the tangent function is the set of all real numbers.
92. Show that the range of the cotangent function is the set of all real numbers.

Discussion and Writing
100. Explain how you would find the value of \( \sin 390^\circ \) using periodic properties.
101. Explain how you would find the value of \( \cos(-45^\circ) \) using even–odd properties.
102. Write down five properties of the tangent function. Explain the meaning of each.
103. Describe your understanding of the meaning of a periodic function.

‘Are You Prepared?’ Answers
1. \( x^2 + y^2 = 1 \)  
2. \( \{x | x \neq 4\} \)  
3. even
7.6 Graphs of the Sine and Cosine Functions*

**PREPARING FOR THIS SECTION**

*Before getting started, review the following:

- **Graphing Techniques: Transformations** (Section 3.5, pp. 252–260)

**New Work** the ‘Are You Prepared?’ problems on page 570.

**OBJECTIVES**

1. Graph Functions of the Form \( y = A \sin (\omega x) \) Using Transformations (p. 561)
2. Graph Functions of the Form \( y = A \cos (\omega x) \) Using Transformations (p. 563)
3. Determine the Amplitude and Period of Sinusoidal Functions (p. 564)
4. Graph Sinusoidal Functions Using Key Points (p. 565)
5. Find an Equation for a Sinusoidal Graph (p. 569)

Since we want to graph the trigonometric functions in the xy-plane, we shall use the traditional symbols \( x \) for the independent variable (or argument) and \( y \) for the dependent variable (or value at \( x \)) for each function. So we write the six trigonometric functions as

\[
\begin{align*}
y = f(x) &= \sin x & y = f(x) &= \cos x & y = f(x) &= \tan x \\
y = f(x) &= \csc x & y = f(x) &= \sec x & y = f(x) &= \cot x
\end{align*}
\]

Here the independent variable \( x \) represents an angle, measured in radians. In calculus, \( x \) will usually be treated as a real number. As we said earlier, these are equivalent ways of viewing \( x \).

**The Graph of the Sine Function \( y = \sin x \)**

Since the sine function has period \( 2\pi \), we need to graph \( y = \sin x \) only on the interval \([0, 2\pi] \). The remainder of the graph will consist of repetitions of this portion of the graph.

We begin by constructing Table 7, which lists some points on the graph of \( y = \sin x \), \( 0 \leq x \leq 2\pi \). As the table shows, the graph of \( y = \sin x \), \( 0 \leq x \leq 2\pi \), begins at the origin. As \( x \) increases from \( 0 \) to \( \frac{\pi}{2} \), the value of \( y = \sin x \) increases from 0 to 1; as \( x \) increases from \( \frac{\pi}{2} \) to \( \pi \) to \( \frac{3\pi}{2} \), the value of \( y \) decreases from 1 to 0 to \(-1\); as \( x \) increases from \( \frac{3\pi}{2} \) to \( 2\pi \), the value of \( y \) increases from \(-1\) to 0. If we plot the points listed in Table 7 and connect them with a smooth curve, we obtain the graph shown in Figure 74.

![Figure 74](image)

The graph in Figure 74 is one period, or **cycle**, of the graph of \( y = \sin x \). To obtain a more complete graph of \( y = \sin x \), we continue the graph in each direction, as shown in Figure 75.

* For those who wish to include phase shifts here, Section 7.8 can be covered immediately after Section 7.6 without loss of continuity.
The graph of \( y = \sin x \) illustrates some of the facts that we already know about the sine function.

**Properties of the Sine Function \( y = \sin x \)**

1. The domain is the set of all real numbers.
2. The range consists of all real numbers from \(-1\) to \(1\), inclusive.
3. The sine function is an odd function, as the symmetry of the graph with respect to the origin indicates.
4. The sine function is periodic, with period \(2\pi\).
5. The \( x \)-intercepts are \(\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots\); the \( y \)-intercept is 0.
6. The maximum value is 1 and occurs at \( x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\)
   the minimum value is \(-1\) and occurs at \( x = \ldots, -\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, -\frac{7\pi}{2}, \ldots\).

**Graph Functions of the Form \( y = A \sin(\omega x) \)**

**EXAMPLE 1**

Graphing Functions of the Form \( y = A \sin(\omega x) \) Using Transformations

Graph \( y = 3 \sin x \) using transformations.

**Solution**

Figure 76 illustrates the steps.
EXAMPLE 2  
Graphing Functions of the Form $y = A \sin(\omega x)$ Using Transformations

Graph $y = -\sin(2x)$ using transformations.

Solution  
Figure 77 illustrates the steps.

Notice in Figure 77(c) that the period of the function $y = -\sin(2x)$ is $\pi$ due to the horizontal compression of the original period $2\pi$ by a factor of $\frac{1}{2}$.

Now Work  

Problem 45 Using Transformations

The Graph of the Cosine Function

The cosine function also has period $2\pi$. We proceed as we did with the sine function by constructing Table 8, which lists some points on the graph of $y = \cos x$, $0 \leq x \leq 2\pi$. As the table shows, the graph of $y = \cos x$, $0 \leq x \leq 2\pi$, begins at the point $(0, 1)$. As $x$ increases from 0 to $\frac{\pi}{2}$, the value of $y$ decreases from 1 to 0 to $-1$; as $x$ increases from $\pi$ to $\frac{3\pi}{2}$ to $2\pi$, the value of $y$ increases from $-1$ to 0 to 1. As before, we plot the points in Table 8 to get one period or cycle of the graph. See Figure 78.

A more complete graph of $y = \cos x$ is obtained by continuing the graph in each direction, as shown in Figure 79.

The graph of $y = \cos x$ illustrates some of the facts that we already know about the cosine function.
**Properties of the Cosine Function**

1. The domain is the set of all real numbers.
2. The range consists of all real numbers from \(-1\) to 1, inclusive.
3. The cosine function is an even function, as the symmetry of the graph with respect to the \(y\)-axis indicates.
4. The cosine function is periodic, with period \(2\pi\).
5. The \(x\)-intercepts are \(\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\); the \(y\)-intercept is 1.
6. The maximum value is 1 and occurs at \(x = \ldots, -\pi, 0, \pi, 3\pi, 5\pi, \ldots\); the minimum value is \(-1\) and occurs at \(x = \ldots, -\pi, \pi, 3\pi, 5\pi, \ldots\).

## 2. Graph Functions of the Form \(y = A \cos(\omega x)\)

### Using Transformations

**EXAMPLE 3**

Graphing Functions of the Form \(y = A \cos(\omega x)\)

**Using Transformations**

Graph \(y = 2 \cos(3x)\) using transformations.

**Solution**

Figure 80 shows the steps.

Notice in Figure 80(c) that the period of the function \(y = 2 \cos(3x)\) is \(\frac{2\pi}{3}\) due to the compression of the original period \(2\pi\) by a factor of \(\frac{1}{3}\).

**Now Work**

**Problem 53 Using Transformations**

**Sinusoidal Graphs**

Shift the graph of \(y = \cos x\) to the right \(\frac{\pi}{2}\) units to obtain the graph of \(y = \cos\left(x - \frac{\pi}{2}\right)\). See Figure 81(a). Now look at the graph of \(y = \sin x\) in Figure 81(b). We see that the graph of \(y = \sin x\) is the same as the graph of \(y = \cos\left(x - \frac{\pi}{2}\right)\).
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Seeing the Concept
Graph \( Y_1 = \sin x \) and \( Y_2 = \cos \left( x - \frac{\pi}{2} \right) \). How many graphs do you see?

Based on Figure 81, we conjecture that

\[
\sin x = \cos \left( x - \frac{\pi}{2} \right)
\]

(We shall prove this fact in Chapter 8.) Because of this relationship, the graphs of functions of the form \( y = A \sin(\omega x) \) or \( y = A \cos(\omega x) \) are referred to as sinusoidal graphs.

Let’s look at some general properties of sinusoidal graphs.

3 Determine the Amplitude and Period of Sinusoidal Functions
In Figure 82(b) we show the graph of \( y = 2 \cos x \). Notice that the values of \( y = 2 \cos x \) lie between \(-2\) and \(2\), inclusive.

In general, the values of the functions \( y = A \sin x \) and \( y = A \cos x \), where \( A \neq 0 \), will always satisfy the inequalities

\[
-|A| \leq A \sin x \leq |A| \quad \text{and} \quad -|A| \leq A \cos x \leq |A|
\]

respectively. The number \( |A| \) is called the amplitude of \( y = A \sin x \) or \( y = A \cos x \). See Figure 83.

In Figure 84(b), we show the graph of \( y = \cos(3x) \). Notice that the period of this function is \( \frac{2\pi}{3} \), due to the horizontal compression of the original period \( 2\pi \) by a factor of \( \frac{1}{3} \).

In general, if \( \omega > 0 \), the functions \( y = \sin(\omega x) \) and \( y = \cos(\omega x) \) will have period \( T = \frac{2\pi}{\omega} \). To see why, recall that the graph of \( y = \sin(\omega x) \) is obtained from the
graph of \( y = \sin x \) by performing a horizontal compression or stretch by a factor \( \frac{1}{\omega} \).

This horizontal compression replaces the interval \([0, 2\pi]\), which contains one period of the graph of \( y = \sin x \), by the interval \([0, \frac{2\pi}{\omega}]\), which contains one period of the graph of \( y = \sin(\omega x) \). The period of the functions \( y = \sin(\omega x) \) and \( y = \cos(\omega x) \), \( \omega > 0 \), is \( \frac{2\pi}{\omega} \).

For example, for the function \( y = \cos(3x) \), graphed in Figure 84(b), \( \omega = 3 \), so the period is \( \frac{2\pi}{\omega} = \frac{2\pi}{3} \).

One period of the graph of \( y = \sin(\omega x) \) or \( y = \cos(\omega x) \) is called a cycle. Figure 85 illustrates the general situation. The blue portion of the graph is one cycle.

![Figure 85](image)

If \( \omega < 0 \) in \( y = \sin(\omega x) \) or \( y = \cos(\omega x) \), we use the Even–Odd Properties of the sine and cosine functions as follows:

\[
\sin(-\omega x) = -\sin(\omega x) \quad \text{and} \quad \cos(-\omega x) = \cos(\omega x)
\]

This gives us an equivalent form in which the coefficient of \( x \) in the argument is positive. For example,

\[
\sin(-2x) = -\sin(2x) \quad \text{and} \quad \cos(-\pi x) = \cos(\pi x)
\]

Because of this, we can assume \( \omega > 0 \).

**Theorem**

If \( \omega > 0 \), the amplitude and period of \( y = A \sin(\omega x) \) and \( y = A \cos(\omega x) \) are given by

\[
\text{Amplitude} = |A| \quad \text{Period} = \frac{2\pi}{\omega}
\]

**Example 4**

**Finding the Amplitude and Period of a Sinusoidal Function**

Determine the amplitude and period of \( y = 3 \sin(4x) \).

**Solution**

Comparing \( y = 3 \sin(4x) \) to \( y = A \sin(\omega x) \), we find that \( A = 3 \) and \( \omega = 4 \). From equation (1),

\[
\text{Amplitude} = |A| = 3 \quad \text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}
\]

**Now Work**

**Problem 23**

**Section 7.6 Graphs of the Sine and Cosine Functions**

**Graph Sinusoidal Functions Using Key Points**

So far, we have graphed functions of the form \( y = A \sin(\omega x) \) or \( y = A \cos(\omega x) \) using transformations. We now introduce another method that can be used to graph these functions.
Figure 86 shows one cycle of the graphs of \( y = \sin x \) and \( y = \cos x \) on the interval \([0, 2\pi]\). Notice that each graph consists of four parts corresponding to the four subintervals:
\[
\left[0, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \pi\right], \left[\pi, \frac{3\pi}{2}\right], \left[\frac{3\pi}{2}, 2\pi\right]
\]
Each subinterval is of length \( \frac{\pi}{2} \) (the period \( 2\pi \) divided by 4, the number of parts), and the endpoints of these intervals \( x = 0, x = \frac{\pi}{2}, x = \pi, x = \frac{3\pi}{2}, x = 2\pi \) give rise to five key points on each graph:

For \( y = \sin x \):
\[
(0, 0), \left(\frac{\pi}{2}, 1\right), (\pi, 0), \left(\frac{3\pi}{2}, -1\right), (2\pi, 0)
\]

For \( y = \cos x \):
\[
(0, 1), \left(\frac{\pi}{2}, 0\right), (\pi, -1), \left(\frac{3\pi}{2}, 0\right), (2\pi, 1)
\]

Look again at Figure 86.

**Steps for Graphing a Sinusoidal Function of the Form**
\( y = A \sin(\omega x) \) or \( y = A \cos(\omega x) \) **Using Key Points**

**Step 1:** Use the amplitude \( A \) to determine the maximum and minimum values of the function. This sets the scale for the \( y \)-axis.

**Step 2:** Use the period \( \frac{2\pi}{\omega} \) and divide the interval \( \left[0, \frac{2\pi}{\omega}\right] \) into four subintervals of the same length.

**Step 3:** Use the endpoints of these subintervals to obtain five key points on the graph.

**Step 4:** Connect these points with a sinusoidal graph to obtain the graph of one cycle and extend the graph in each direction to make it complete.

---

**Example 5**

**Graphing a Sinusoidal Function Using Key Points**

Graph: \( y = 3 \sin(4x) \)

**Solution**

Refer to Example 4. For \( y = 3 \sin(4x) \), the amplitude is 3 and the period is \( \frac{\pi}{2} \). Because the amplitude is 3, the graph of \( y = 3 \sin(4x) \) will lie between \(-3\) and 3 on the \( y \)-axis. Because the period is \( \frac{\pi}{2} \), one cycle will begin at \( x = 0 \) and end at \( x = \frac{\pi}{2} \).

We divide the interval \( \left[0, \frac{\pi}{2}\right] \) into four subintervals, each of length \( \frac{\pi}{8} \), by finding the following values:

\[
\begin{array}{cccccc}
0 & 0 & + & \frac{\pi}{8} & = & \frac{\pi}{8} \\
& & & & & \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4} \\
& & & & & \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4} \\
& & & & & \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{2} \\
\end{array}
\]

Initial value 2nd value 3rd value 4th value final value
These values of $x$ determine the $x$-coordinates of the five key points on the graph. To obtain the $y$-coordinates of the five key points of $y = 3 \sin(4x)$, we multiply the $y$-coordinates of the five key points for $y = \sin x$ in Figure 86(a) by $A = 3$. The five key points are

$$
(0, 0), \quad \left(\frac{\pi}{8}, 3\right), \quad \left(\frac{\pi}{4}, 0\right), \quad \left(\frac{3\pi}{8}, -3\right), \quad \left(\frac{\pi}{2}, 0\right)
$$

We plot these five points and fill in the graph of the sine curve as shown in Figure 87(a). We extend the graph in either direction to obtain the complete graph shown in Figure 87(b).

**Check:** Graph $y = 3 \sin(4x)$ using transformations. Which graphing method do you prefer?

**Now Work** PROBLEM 45 USING KEY POINTS

**EXAMPLE 6** Finding the Amplitude and Period of a Sinusoidal Function and Graphing It Using Key Points

Determine the amplitude and period of $y = 2 \sin\left(\frac{\pi}{2} x\right)$, and graph the function.

**Solution** Since the sine function is odd, we can use the equivalent form:

$$
y = -2 \sin\left(\frac{\pi}{2} x\right)
$$

Comparing $y = -2 \sin\left(\frac{\pi}{2} x\right)$ to $y = A \sin(\omega x)$, we find that $A = -2$ and $\omega = \frac{\pi}{2}$.

The amplitude is $|A| = 2$, and the period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/2} = 4$.

The graph of $y = -2 \sin\left(\frac{\pi}{2} x\right)$ will lie between $-2$ and $2$ on the $y$-axis. One cycle will begin at $x = 0$ and end at $x = 4$. We divide the interval $[0, 4]$ into four subintervals, each of length $4 \div 4 = 1$, by finding the following values:

<table>
<thead>
<tr>
<th>Initial value</th>
<th>1st value</th>
<th>2nd value</th>
<th>3rd value</th>
<th>Final value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 + 1 = 1</td>
<td>1 + 1 = 2</td>
<td>2 + 1 = 3</td>
<td>3 + 1 = 4</td>
</tr>
</tbody>
</table>

Since $y = -2 \sin\left(\frac{\pi}{2} x\right)$, we multiply the $y$-coordinates of the five key points in Figure 86(a) by $A = -2$. The five key points on the graph are

$$
(0, 0), \quad (-1, -2), \quad (2, 0), \quad (3, 2), \quad (4, 0)
$$

We plot these five points and fill in the graph of the sine function as shown in Figure 88(a) on page 568. Extending the graph in each direction, we obtain Figure 88(b).
CHAPTER 7  Trigonometric Functions

Check: Graph \( y = 2 \sin \left(-\frac{\pi}{2} x\right) \) using transformations. Which graphing method do you prefer?

Now Work PROBLEM 49 USING KEY POINTS

If the function to be graphed is of the form \( y = A \sin(\omega x) + B \) [or \( y = A \cos(\omega x) + B \)], first graph \( y = A \sin(\omega x) \) [or \( y = A \cos(\omega x) \)] and then use a vertical shift.

**Example 7** Finding the Amplitude and Period of a Sinusoidal Function and Graphing It Using Key Points

Determine the amplitude and period of \( y = -4 \cos(\pi x) - 2 \), and graph the function.

**Solution**

We begin by graphing the function \( y = -4 \cos(\pi x) \). Comparing \( y = -4 \cos(\pi x) \) with \( y = A \cos(\omega x) \), we find that \( A = -4 \) and \( \omega = \pi \). The amplitude is \( |A| = |-4| = 4 \), and the period is \( T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2 \).

The graph of \( y = -4 \cos(\pi x) \) will lie between \(-4\) and \(4\) on the \(y\)-axis. One cycle will begin at \( x = 0 \) and end at \( x = 2 \). We divide the interval \([0, 2]\) into four subintervals, each of length \( \frac{2}{4} = \frac{1}{2} \), by finding the following values:

\[
\begin{array}{cccccc}
0 & 0 + \frac{1}{2} & \frac{1}{2} & 1 + \frac{1}{2} & 3 + \frac{1}{2} & 2 \\
nitial value & 1st value & 2nd value & 3rd value & final value
\end{array}
\]

Since \( y = -4 \cos(\pi x) \), we multiply the \(y\)-coordinates of the five key points of \( y = \cos x \) shown in Figure 86(b) by \(-4\) to obtain the five key points on the graph of \( y = -4 \cos(\pi x) \):

\[
(0, -4), \left(\frac{1}{2}, 0\right), \left(1, 4\right), \left(\frac{3}{2}, 0\right), \left(2, -4\right)
\]

We plot these five points and fill in the graph of the cosine function as shown in Figure 89(a). Extending the graph in each direction, we obtain Figure 89(b), the graph of \( y = -4 \cos(\pi x) \). A vertical shift down \(2\) units gives the graph of \( y = -4 \cos(\pi x) - 2 \), as shown in Figure 89(c).
5 Find an Equation for a Sinusoidal Graph

We can also use the ideas of amplitude and period to identify a sinusoidal function when its graph is given.

**EXAMPLE 8** Finding an Equation for a Sinusoidal Graph

Find an equation for the graph shown in Figure 90.

![Figure 90](image)

**Solution** The graph has the characteristics of a cosine function. Do you see why? So we view the equation as a cosine function with period $T = 1$. Then $\frac{2\pi}{\omega} = 1$, so $\omega = 2\pi$. The cosine function whose graph is given in Figure 90 is

$$y = A \cos(\omega x) = 3 \cos(2\pi x)$$

**Check:** Graph $Y_1 = 3 \cos(2\pi x)$ and compare the result with Figure 90.

**EXAMPLE 9** Finding an Equation for a Sinusoidal Graph

Find an equation for the graph shown in Figure 91.

![Figure 91](image)

**Solution** The graph is sinusoidal, with amplitude $|A| = 2$. The period is 4, so $\frac{2\pi}{\omega} = 4$ or $\omega = \frac{\pi}{2}$. Since the graph passes through the origin, it is easiest to view the equation as a sine function, but notice that the graph is actually the reflection of a sine function about the $x$-axis (since the graph is decreasing near the origin). This requires that $A = -2$. The sine function whose graph is given in Figure 91 is

$$y = A \sin(\omega x) = -2 \sin\left(\frac{\pi}{2} x\right)$$

**Check:** Graph $Y_1 = -2 \sin\left(\frac{\pi}{2} x\right)$ and compare the result with Figure 91.

* The equation could also be viewed as a cosine function with a horizontal shift, but viewing it as a sine function is easier.
7.6 Assess Your Understanding

‘Are You Prepared?’ Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Use transformations to graph \( y = 3x^2 \) (pp. 252–260)
2. Use transformations to graph \( y = -x^2 \) (pp. 252–260)

Concepts and Vocabulary

3. The maximum value of \( y = \sin x, 0 \leq x \leq 2\pi \), is ____ and occurs at \( x = \) ____.
4. The function \( y = A \sin(\omega x) \), \( A > 0 \), has amplitude 3 and period 2; then \( A = \) ____ and \( \omega = \) ____.
5. The function \( y = 3 \cos(6x) \) has amplitude ____ and period ____.

6. True or False The graphs of \( y = \sin x \) and \( y = \cos x \) are identical except for a horizontal shift.
7. True or False For \( y = 2 \sin(\pi x) \), the amplitude is 2 and the period is \( \pi/2 \).
8. True or False The graph of the sine function has infinitely many \( x \)-intercepts.

Skill Building

In Problems 9–18, if necessary, refer to a graph to answer each question.

9. What is the \( y \)-intercept of \( y = \sin x \)?
10. What is the \( y \)-intercept of \( y = \cos x \)?
11. For what numbers \( x, -\pi \leq x \leq \pi \), is the graph of \( y = \sin x \) increasing?
12. For what numbers \( x, -\pi \leq x \leq \pi \), is the graph of \( y = \cos x \) decreasing?
13. What is the largest value of \( y = \sin x \)?
14. What is the smallest value of \( y = \cos x \)?
15. For what numbers \( x, 0 \leq x \leq 2\pi \), does \( \sin x = 0 \)?
16. For what numbers \( x, 0 \leq x \leq 2\pi \), does \( \cos x = 0 \)?
17. For what numbers \( x, -2\pi \leq x \leq 2\pi \), does \( \sin x = 1 \)? Where does \( \sin x = -1 \)?
18. For what numbers \( x, -2\pi \leq x \leq 2\pi \), does \( \cos x = 1 \)? Where does \( \cos x = -1 \)?

In Problems 19–28, determine the amplitude and period of each function without graphing.

19. \( y = 2 \sin x \)
20. \( y = 3 \cos x \)
21. \( y = -4 \cos(2x) \)
22. \( y = -\sin\left(\frac{1}{2}x\right)\)
23. \( y = 6 \sin(\pi x) \)
24. \( y = -3 \cos(3x) \)
25. \( y = -\frac{1}{2} \cos\left(\frac{3}{2}x\right) \)
26. \( y = \frac{4}{3} \sin\left(\frac{2}{3}x\right) \)
27. \( y = \frac{5}{3} \sin\left(-\frac{2\pi}{3} x\right) \)
28. \( y = \frac{9}{5} \cos\left(-\frac{3\pi}{2} x\right) \)

In Problems 29–38, match the given function to one of the graphs (A)–(J).

\[
\begin{array}{ccc}
\text{(A)} & \text{(B)} & \text{(C)} \\
\text{(D)} & \text{(E)} & \text{(F)} \\
\text{(G)} & \text{(H)} & \text{(I)} \\
\end{array}
\]
In Problems 39–42, match the given function to one of the graphs (A)–(D).

39. \( y = 3 \sin \left( \frac{\pi}{2} x \right) \)  
40. \( y = -3 \sin(2x) \)  
41. \( y = 3 \cos(2x) \)  
42. \( y = -3 \sin \left( \frac{1}{2} x \right) \)

In Problems 43–66, graph each function. Be sure to label key points and show at least two cycles.

43. \( y = 4 \cos x \)  
44. \( y = 3 \sin x \)  
45. \( y = -4 \sin x \)  
46. \( y = -3 \cos x \)  
47. \( y = \cos(4x) \)  
48. \( y = \sin(3x) \)  
49. \( y = \sin(-2x) \)  
50. \( y = \cos(-2x) \)  
51. \( y = 2 \sin \left( \frac{1}{2} x \right) \)  
52. \( y = 2 \cos \left( \frac{1}{4} x \right) \)  
53. \( y = -\frac{1}{2} \cos(2x) \)  
54. \( y = -4 \sin \left( \frac{1}{8} x \right) \)  
55. \( y = 2 \sin x + 3 \)  
56. \( y = 3 \cos x + 2 \)  
57. \( y = 5 \cos(\pi x) - 3 \)  
58. \( y = 4 \sin \left( \frac{\pi}{2} x \right) - 2 \)  
59. \( y = -6 \sin \left( \frac{\pi}{3} x \right) + 4 \)  
60. \( y = -3 \cos \left( \frac{\pi}{4} x \right) + 2 \)  
61. \( y = 5 - 3 \sin(2x) \)  
62. \( y = 2 - 4 \cos(3x) \)  
63. \( y = \frac{5}{3} \sin \left( \frac{-2\pi}{3} x \right) \)  
64. \( y = \frac{9}{5} \cos \left( -\frac{3\pi}{2} x \right) \)  
65. \( y = -\frac{3}{2} \cos \left( \frac{\pi}{4} x \right) + \frac{1}{2} \)  
66. \( y = -\frac{1}{2} \sin \left( \frac{\pi}{8} x \right) + \frac{3}{2} \)

In Problems 67–70, write the equation of a sine function that has the given characteristics.

67. Amplitude: 3  
Period: \( \pi \)

68. Amplitude: 2  
Period: \( 4\pi \)

69. Amplitude: 3  
Period: \( 2 \pi \)

70. Amplitude: 4  
Period: \( 1 \)

In Problems 71–84, find an equation for each graph.
Applications and Extensions

In Problems 85–88, find the average rate of change of \( f \) from 0 to \( \frac{\pi}{2} \).

85. \( f(x) = \sin x \)
86. \( f(x) = \cos x \)
87. \( f(x) = \sin \left( \frac{x}{2} \right) \)
88. \( f(x) = \cos(2x) \)

In Problems 89–92, find \((f \circ g)(x)\) and \((g \circ f)(x)\) and graph each of these functions.

89. \( f(x) = \sin x \)
   \( g(x) = 4x \)
90. \( f(x) = \cos x \)
   \( g(x) = \frac{1}{2}x \)
91. \( f(x) = -2x \)
   \( g(x) = \cos x \)
92. \( f(x) = -3x \)
   \( g(x) = \sin x \)

93. **Alternating Current (ac) Circuits** The current \( I \), in amperes, flowing through an ac (alternating current) circuit at time \( t \) in seconds, is

\[
I(t) = 220 \sin(60\pi t) \quad t \geq 0
\]

What is the period? What is the amplitude? Graph this function over two periods.

94. **Alternating Current (ac) Circuits** The current \( I \), in amperes, flowing through an ac (alternating current) circuit at time \( t \) in seconds, is

\[
I(t) = 120 \sin(30\pi t) \quad t \geq 0
\]

What is the period? What is the amplitude? Graph this function over two periods.

95. **Alternating Current (ac) Generators** The voltage \( V \), in volts, produced by an ac generator at time \( t \), in seconds, is

\[
V(t) = 220 \sin(120\pi t)
\]

(a) What is the amplitude? What is the period?
(b) Graph \( V \) over two periods, beginning at \( t = 0 \).

96. **Alternating Current (ac) Generators** The voltage \( V \), in volts, produced by an ac generator at time \( t \), in seconds, is

\[
V(t) = 120 \sin(120\pi t)
\]

(a) What is the amplitude? What is the period?
(b) Graph \( V \) over two periods, beginning at \( t = 0 \).
(c) If a resistance of \( R \) = 10 ohms is present, what is the current \( I \)?
   [Hint: Use Ohm’s Law, \( V = IR \).]
(d) What is the amplitude and period of the current \( I \)?
(e) Graph \( I \) over two periods, beginning at \( t = 0 \).
97. **Alternating Current (ac) Generators** The voltage $V$ produced by an ac generator is sinusoidal. As a function of time, the voltage $V$ is

$$ V(t) = V_0 \sin(2\pi ft) $$

where $f$ is the **frequency**, the number of complete oscillations (cycles) per second. [In the United States and Canada, $f$ is 60 hertz (Hz).] The **power** $P$ delivered to a resistance $R$ at any time $t$ is defined as

$$ P(t) = \frac{(V(t))^2}{R} $$

(a) Show that $P(t) = \frac{V_0^2}{R} \sin^2(2\pi ft)$.

(b) The graph of $P$ is shown in the figure. Express $P$ as a sinusoidal function.

![Power in an ac generator](image)

(c) Deduce that

$$ \sin^2(2\pi ft) = \frac{1}{2} [1 - \cos(4\pi ft)] $$

98. **Bridge Clearance** A one-lane highway runs through a tunnel in the shape of one-half a sine curve cycle. The opening is 28 feet wide at road level and is 15 feet tall at its highest point.

![Bridge Clearance](image)

99. **Biorhythms** In the theory of biorhythms, a sine function of the form $P(t) = 50 \sin(\omega t) + 50$ is used to measure the percent $P$ of a person’s potential at time $t$, where $t$ is measured in days and $t = 0$ is the person’s birthday. Three characteristics are commonly measured:

- Physical potential: period of 23 days
- Emotional potential: period of 28 days
- Intellectual potential: period of 33 days

(a) Find $\omega$ for each characteristic.

(b) Using a graphing utility, graph all three functions on the same screen.

(c) Is there a time $t$ when all three characteristics have 100% potential? When is it?

(d) Suppose that you are 20 years old today. Describe your physical, emotional, and intellectual potential for the next 30 days.

Discussion and Writing

102. Explain how you would scale the $x$-axis and $y$-axis before graphing $y = 3 \cos(\pi x)$.

103. Explain the term **amplitude** as it relates to the graph of a sinusoidal function.

'Are You Prepared?' Answers

1. Vertical stretch by a factor of 3

![Vertical stretch by a factor of 3](image)

2. Reflection about the $x$-axis

![Reflection about the x-axis](image)
The Graph of the Tangent Function

Because the tangent function has period $\pi$, we only need to determine the graph over some interval of length $\pi$. The rest of the graph will consist of repetitions of that graph. Because the tangent function is not defined at $\pm \frac{\pi}{2}$, we will concentrate on the interval of length $\pi$, and construct Table 9, which lists some points on the graph of $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. We plot the points in the table and connect them with a smooth curve. See Figure 92 for a partial graph of where $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = \tan x$</th>
<th>$(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{\pi}{3}$</td>
<td>$-\sqrt{3} \approx -1.73$</td>
<td>$(-\frac{\pi}{3}, -\sqrt{3})$</td>
</tr>
<tr>
<td>$-\frac{\pi}{4}$</td>
<td>$-1$</td>
<td>$(-\frac{\pi}{4}, -1)$</td>
</tr>
<tr>
<td>$-\frac{\pi}{6}$</td>
<td>$-\frac{\sqrt{3}}{3} \approx -0.58$</td>
<td>$(-\frac{\pi}{6}, -\frac{\sqrt{3}}{3})$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$(0, 0)$</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\sqrt{3}}{3} \approx 0.58$</td>
<td>$(\frac{\pi}{6}, \frac{\sqrt{3}}{3})$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>1</td>
<td>$(\frac{\pi}{4}, 1)$</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$\sqrt{3} \approx 1.73$</td>
<td>$(\frac{\pi}{3}, \sqrt{3})$</td>
</tr>
</tbody>
</table>

To complete one period of the graph of $y = \tan x$, we need to investigate the behavior of the function as $x$ approaches $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. We must be careful, though, because $y = \tan x$ is not defined at these numbers. To determine this behavior, we use the identity

$$\tan x = \frac{\sin x}{\cos x}$$

See Table 10. If $x$ is close to $\frac{\pi}{2} \approx 1.5708$, but remains less than $\frac{\pi}{2}$, then $\sin x$ will be close to 1 and $\cos x$ will be positive and close to 0. (To see this, refer back to the
graphs of the sine function and the cosine function.) So the ratio \( \frac{\sin x}{\cos x} \) will be pos-
itive and large. In fact, the closer \( x \) gets to \( \frac{\pi}{2} \), the closer \( \sin x \) gets to 1 and \( \cos x \) gets
to 0, so \( \tan x \) approaches \( \infty \left( \lim_{x \to \frac{\pi}{2}} \tan x = \infty \right) \). In other words, the vertical line
\( x = \frac{\pi}{2} \) is a vertical asymptote to the graph of \( y = \tan x \).

If \( x \) is close to \(-\frac{\pi}{2}\), but remains greater than \(-\frac{\pi}{2}\), then \( \sin x \) will be close to \(-1\) and
\( \cos x \) will be positive and close to 0. The ratio \( \frac{\sin x}{\cos x} \) approaches
\( -\infty \left( \lim_{x \to -\frac{\pi}{2}} \tan x = -\infty \right) \). In other words, the vertical line \( x = -\frac{\pi}{2} \) is also a vertical
asymptote to the graph.

With these observations, we can complete one period of the graph. We obtain
the complete graph of \( y = \tan x \) by repeating this period, as shown in Figure 93.

The graph of \( y = \tan x \) in Figure 93 illustrates the following properties.

**Properties of the Tangent Function**

1. The domain is the set of all real numbers, except odd multiples of \( \frac{\pi}{2} \).
2. The range is the set of all real numbers.
3. The tangent function is an odd function, as the symmetry of the graph
with respect to the origin indicates.

(continued)
4. The tangent function is periodic, with period $\pi$.
5. The $x$-intercepts are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$; the $y$-intercept is 0.
6. Vertical asymptotes occur at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \ldots$.

**EXAMPLE 1**

**Graphing Functions of the Form $y = A \tan(\omega x) + B$**

Graph: $y = 2 \tan x - 1$

**Solution**

Figure 94 shows the steps using transformations.

**EXAMPLE 2**

**Graphing Functions of the Form $y = A \tan(\omega x) + B$**

Graph: $y = 3 \tan(2x)$

**Solution**

Figure 95 shows the steps using transformations.
Notice in Figure 95(c) that the period of \( y = 3 \tan(2x) \) is \( \frac{\pi}{2} \) due to the compression of the original period \( \pi \) by a factor of \( \frac{1}{2} \). Also notice that the asymptotes are \( x = -\frac{\pi}{4}, x = \frac{\pi}{4}, x = \frac{3\pi}{4} \), and so on, also due to the compression.

### The Graph of the Cotangent Function

We obtain the graph of \( y = \cot x \) as we did the graph of \( y = \tan x \). The period of \( y = \cot x \) is \( \pi \). Because the cotangent function is not defined for integer multiples of \( \pi \), we will concentrate on the interval \( (0, \pi) \). Table 11 lists some points on the graph of \( y = \cot x \).

As \( x \) approaches 0, but remains greater than 0, the value of \( \cos x \) will be close to 1 and the value of \( \sin x \) will be positive and close to 0. Hence, the ratio will be positive and large; so as \( x \) approaches 0, with \( x \) approaches \( \infty \). Similarly, as \( x \) approaches \( \pi \), but remains less than \( \pi \), the value of \( \cos x \) will be close to -1, and the value of \( \sin x \) will be positive and close to 0. So the ratio \( \frac{\cos x}{\sin x} = \cot x \) will be negative and will approach \( -\infty \) as \( x \) approaches \( \pi \) \( \lim_{x \to \pi^-} \cot x = -\infty \). Figure 96 shows the graph.

### The Graph of the Cosecant Function and the Secant Function

The cosecant and secant functions, sometimes referred to as reciprocal functions, are graphed by making use of the reciprocal identities

\[
\csc x = \frac{1}{\sin x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}
\]
For example, the value of the cosecant function \( y = \csc x \) at a given number \( x \) equals the reciprocal of the corresponding value of the sine function, provided that the value of the sine function is not 0. If the value of \( \sin x \) is 0, then \( x \) is an integer multiple of \( \pi \). At such numbers, the cosecant function is not defined. In fact, the graph of the cosecant function has vertical asymptotes at integer multiples of \( \pi \). Figure 97 shows the graph.

**Figure 97**
\( y = \csc x, -\infty < x < \infty, x \) not equal to integer multiples of \( \pi, |y| \geq 1 \)

Using the idea of reciprocals, we can similarly obtain the graph of \( y = \sec x \). See Figure 98.

**Figure 98**
\( y = \sec x, -\infty < x < \infty, x \) not equal to odd multiples of \( \pi, \frac{\pi}{2}, |y| \geq 1 \)

### 2 Graph Functions of the Form \( y = A \csc(\omega x) + B \) and \( y = A \sec(\omega x) + B \)

The role of \( A \) in these functions is to set the range. The range of \( y = \csc x \) is \( \{y \mid |y| \geq 1\} \); the range of \( y = A \csc x \) is \( \{y \mid |y| \geq |A|\} \), due to the vertical stretch of the graph by a factor of \( |A| \). Just as with the sine and cosine functions, the period of \( y = \csc(\omega x) \) and \( y = \sec(\omega x) \) becomes \( \frac{2\pi}{\omega} \) due to the horizontal compression of the graph by a factor of \( \frac{1}{\omega} \). The presence of \( B \) indicates a vertical shift is required.

We shall graph these functions in two ways: using transformations and using the reciprocal function.

**EXAMPLE 3** Graphing Functions of the Form \( y = A \csc(\omega x) + B \)

Graph: \( y = 2 \csc x - 1 \)
Solution Using Transformations

Figure 99 shows the required steps.

Solution Using the Reciprocal Function

We graph $y = 2 \csc x - 1$ by first graphing the reciprocal function $y = 2 \sin x - 1$ and then filling in the graph of $y = 2 \csc x - 1$, using the idea of reciprocals. See Figure 100.

Now Work Problem 29

7.7 Assess Your Understanding

‘Are You Prepared?’ Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. The graph of $y = \frac{3x - 6}{x - 4}$ has a vertical asymptote. What is it? (pp. 346–348)
2. True or False A function $f$ has at most one vertical asymptote. (pp. 346–348)

Concepts and Vocabulary

3. The graph of $y = \tan x$ is symmetric with respect to the _____ and has vertical asymptotes at _____.
4. The graph of $y = \sec x$ is symmetric with respect to the _____ and has vertical asymptotes at _____.
5. It is easiest to graph $y = \sec x$ by first sketching the graph of _____.
6. True or False The graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$ each have infinitely many vertical asymptotes.
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Skill Building

In Problems 7–16, if necessary, refer to the graphs to answer each question.

7. What is the y-intercept of \( y = \tan x \)?
9. What is the y-intercept of \( y = \sec x \)?
11. For what numbers \( x \), \( -2\pi \leq x \leq 2\pi \), does \( \sec x = 1 \)? For what numbers \( x \) does \( \sec x = -1 \)?
13. For what numbers \( x \), \( -2\pi \leq x \leq 2\pi \), does the graph of \( y = \sec x \) have vertical asymptotes?
15. For what numbers \( x \), \( -2\pi \leq x \leq 2\pi \), does the graph of \( y = \tan x \) have vertical asymptotes?

In Problems 17–40, graph each function. Be sure to label key points and show at least two cycles.

17. \( y = 3 \tan x \)
18. \( y = -2 \tan x \)
19. \( y = 4 \cot x \)
20. \( y = -3 \cot x \)
21. \( y = \tan \left( \frac{\pi}{2} x \right) \)
22. \( y = \tan \left( \frac{1}{2} x \right) \)
23. \( y = \cot \left( \frac{1}{4} x \right) \)
24. \( y = \cot \left( \frac{\pi}{4} x \right) \)
25. \( y = 2 \sec x \)
26. \( y = \frac{1}{2} \csc x \)
27. \( y = -3 \csc x \)
28. \( y = -4 \sec x \)
29. \( y = 4 \sec \left( \frac{1}{2} x \right) \)
30. \( y = \frac{1}{2} \csc \left( 2x \right) \)
31. \( y = -2 \csc(\pi x) \)
32. \( y = -3 \csc \left( \frac{\pi}{2} x \right) \)
33. \( y = \tan \left( \frac{1}{4} x \right) + 1 \)
34. \( y = 2 \cot x - 1 \)
35. \( y = \sec \left( \frac{2\pi}{3} x \right) + 2 \)
36. \( y = \csc \left( \frac{3\pi}{2} x \right) \)
37. \( y = \frac{1}{2} \tan \left( \frac{1}{4} x \right) - 2 \)
38. \( y = 3 \cot \left( \frac{1}{2} x \right) - 2 \)
39. \( y = 2 \sec \left( \frac{1}{3} x \right) - 1 \)
40. \( y = 3 \sec \left( \frac{1}{4} x \right) + 1 \)

Applications and Extensions

In Problems 41–44, find the average rate of change of \( f \) from 0 to \( \frac{\pi}{6} \).

41. \( f(x) = \tan x \)
42. \( f(x) = \sec x \)
43. \( f(x) = \tan(2x) \)
44. \( f(x) = \sec(2x) \)

In Problems 45–48, find \( (f \circ g)(x) \) and \( (g \circ f)(x) \) and graph each of these functions.

45. \( f(x) = \tan x \)
\( g(x) = 4x \)
\( f(x) = 2 \sec x \)
\( g(x) = \frac{1}{2} x \)
46. \( f(x) = 2 \sec x \)
\( g(x) = \cot x \)
47. \( f(x) = -2x \)
\( g(x) = \tan x \)
48. \( f(x) = \frac{1}{2} x \)
\( g(x) = 2 \csc x \)

49. Carrying a Ladder around a Corner  Two hallways, one of width 3 feet, the other of width 4 feet, meet at a right angle. See the illustration.
(a) Show that the length \( L \) of the line segment shown as a function of the angle \( \theta \) is
\[ L(\theta) = 3 \sec \theta + 4 \csc \theta \]
(b) Graph \( L = L(\theta) \), \( 0 < \theta < \frac{\pi}{2} \).
(c) For what value of \( \theta \) is \( L \) the least?
(d) What is the length of the longest ladder that can be carried around the corner? Why is this also the least value of \( L \)?

50. A Rotating Beacon  Suppose that a fire truck is parked in front of a building as shown in the figure.

The beacon light on top of the fire truck is located 10 feet from the wall and has a light on each side. If the beacon light rotates 1 revolution every 2 seconds, then a model for determining the distance \( d \) that the beacon of light is from point \( A \) on the wall after \( t \) seconds is given by
\[ d(t) = |10 \tan(\pi t)| \]
(a) Graph \( d(t) = |10 \tan(\pi t)| \) for \( 0 \leq t \leq 2 \).
(b) For what values of \( t \) is the function undefined? Explain what this means in terms of the beam of light on the wall.

(c) Fill in the following table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) = 10 \tan(\pi t) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) Compute \( \frac{d(0.1) - d(0)}{0.1 - 0} \), \( \frac{d(0.2) - d(0.1)}{0.2 - 0.1} \), and so on, for each consecutive value of \( t \). These are called first differences.

(e) Interpret the first differences found in part (d). What is happening to the speed of the beam of light as \( d \) increases?

51. Exploration

Graph

\[ y = \tan(x) \text{ and } y = -\cot \left( x + \frac{\pi}{2} \right) \]

Do you think that \( \tan x = -\cot \left( x + \frac{\pi}{2} \right) \)?
We see that the graph of \( y = A \sin(\omega x - \phi) = A \sin\left(\frac{\phi}{\omega} - \frac{\omega}{\omega}x\right) \) is the same as the graph of \( y = A \sin(\omega x) \), except that it has been shifted \( \frac{\phi}{\omega} \) units (to the right if \( \phi > 0 \) and to the left if \( \phi < 0 \)). This number \( \frac{\phi}{\omega} \) is called the phase shift of the graph of \( y = A \sin(\omega x - \phi) \).

For the graphs of \( y = A \sin(\omega x - \phi) \) or \( y = A \cos(\omega x - \phi) \), \( \omega > 0 \).

### Example 1

**Finding the Amplitude, Period, and Phase Shift of a Sinusoidal Function and Graphing It**

Find the amplitude, period, and phase shift of \( y = 3 \sin(2x - \pi) \) and graph the function.

**Solution**

Comparing

\[
y = 3 \sin(2x - \pi) = 3 \sin\left(2\left(x - \frac{\pi}{2}\right)\right)
\]

to

\[
y = A \sin(\omega x - \phi) = A \sin\left(\phi - \frac{\omega}{\omega}x\right)
\]

we find that \( A = 3 \), \( \omega = 2 \), and \( \phi = \pi \). The graph is a sine curve with amplitude \( |A| = 3 \), period \( T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi \), and phase shift \( \frac{\phi}{\omega} = \frac{\pi}{2} \).

The graph of \( y = 3 \sin(2x - \pi) \) will lie between \(-3\) and \(3\) on the \(y\)-axis. One cycle will begin at \( x = \frac{\phi}{\omega} = \frac{\pi}{2} \) and end at \( x = \frac{\phi}{\omega} + \frac{2\pi}{\omega} = \frac{\pi}{2} + \pi = \frac{3\pi}{2} \). To find the five key points, we divide the interval \( \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \) into four subintervals, each of length \( \pi/4 \), by finding the following values of \( x \):

\[
\begin{align*}
\frac{\pi}{2} & \quad \text{initial value} \\
\frac{\pi}{2} + \frac{\pi}{4} & \quad \text{2nd value} \\
\frac{3\pi}{4} & \quad \text{3rd value} \\
\frac{\pi}{2} + \frac{\pi}{4} & \quad \text{4th value} \\
\frac{5\pi}{4} & \quad \text{final value}
\end{align*}
\]

Use these values of \( x \) to determine the five key points on the graph:

\[
\left(\frac{\pi}{2}, 0\right), \quad \left(\frac{3\pi}{4}, 3\right), \quad \left(\pi, 0\right), \quad \left(\frac{5\pi}{4}, -3\right), \quad \left(\frac{3\pi}{2}, 0\right)
\]

We plot these five points and fill in the graph of the sine function as shown in Figure 103(a). Extending the graph in each direction, we obtain Figure 103(b).
The graph of \( y = 3 \sin(2x - \pi) = 3 \sin \left[2 \left(x - \frac{\pi}{2}\right)\right]\) may also be obtained using transformations. See Figure 104.

**Example 2** Finding the Amplitude, Period, and Phase Shift of a Sinusoidal Function and Graphing It

Find the amplitude, period, and phase shift of \( y = 2 \cos(4x + \pi) + 1 \) and graph the function.

**Solution**

We begin by graphing \( y = 2 \cos(4x + \pi) \). Comparing

\[
y = 2 \cos(4x + \pi) = 2 \cos \left(4 \left(x + \frac{3\pi}{4}\right)\right)
\]

to

\[
y = A \cos(\omega x - \phi) = A \cos \left(\omega \left(x - \frac{\phi}{\omega}\right)\right)
\]

we see that \( A = 2, \omega = 4, \) and \( \phi = -3\pi \). The graph is a cosine curve with amplitude \( |A| = 2 \), period \( T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2} \), and phase shift \( \frac{\phi}{\omega} = -\frac{3\pi}{4} \).

The graph of \( y = 2 \cos(4x + 3\pi) \) will lie between \(-2\) and \(2\) on the \(y\)-axis. One cycle will begin at \( x = \frac{\phi}{\omega} = -\frac{3\pi}{4} \) and end at \( x = \frac{\phi}{\omega} + \frac{2\pi}{\omega} = -\frac{3\pi}{4} + \frac{\pi}{2} = -\frac{\pi}{4} \).

To find the five key points, we divide the interval \([ -\frac{3\pi}{4}, -\frac{\pi}{4} ]\) into four subintervals, each of the length \( \frac{\pi}{2} + 4 = \frac{\pi}{8} \) by finding the following values.

<table>
<thead>
<tr>
<th>Initial value</th>
<th>2nd value</th>
<th>3rd value</th>
<th>4th value</th>
<th>Final value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\frac{3\pi}{4})</td>
<td>(-\frac{3\pi}{8})</td>
<td>(-\frac{5\pi}{8})</td>
<td>(-\frac{5\pi}{8})</td>
<td>(-\frac{3\pi}{8})</td>
</tr>
</tbody>
</table>

The five key points on the graph of \( y = 2 \cos(4x + \pi) \) are

\((-\frac{3\pi}{4}, 2), \ (-\frac{5\pi}{8}, 0), \ (-\frac{\pi}{2}, -2), \ (-\frac{3\pi}{8}, 0), \ (-\frac{\pi}{4}, 2)\)

We plot these five points and fill in the graph of the cosine function as shown in Figure 105(a). Extending the graph in each direction, we obtain Figure 105(b), the graph of \( y = 2 \cos(4x + \pi) \). A vertical shift up 1 unit gives the final graph. See Figure 105(c).
**SUMMARY**  Steps for Graphing Sinusoidal Functions $y = A \sin(\omega x - \phi) + B$ or $y = A \cos(\omega x - \phi) + B$

**Step 1:** Determine the amplitude $|A|$ and period $T = \frac{2\pi}{\omega}$.

**Step 2:** Determine the starting point of one cycle of the graph, $\frac{\phi}{\omega}$.

**Step 3:** Determine the ending point of one cycle of the graph, $\frac{\phi}{\omega} + \frac{2\pi}{\omega}$.

**Step 4:** Divide the interval $\left[\frac{\phi}{\omega}, \frac{\phi}{\omega} + \frac{2\pi}{\omega}\right]$ into four subintervals, each of length $\frac{2\pi}{\omega} / 4$.

**Step 5:** Use the endpoints of the subintervals to find the five key points on the graph.

**Step 6:** Fill in one cycle of the graph.

**Step 7:** Extend the graph in each direction to make it complete.

**Step 8:** If $B \neq 0$, apply a vertical shift.
2 Find a Sinusoidal Function from Data

Scatter diagrams of data sometimes take the form of a sinusoidal function. Let’s look at an example.

The data given in Table 12 represent the average monthly temperatures in Denver, Colorado. Since the data represent average monthly temperatures collected over many years, the data will not vary much from year to year and so will essentially repeat each year. In other words, the data are periodic. Figure 107 shows the scatter diagram of these data repeated over 2 years, where \( x = 2 \) represents February, and so on.

### Table 12

<table>
<thead>
<tr>
<th>Month, x</th>
<th>Average Monthly Temperature, °F</th>
</tr>
</thead>
<tbody>
<tr>
<td>January, 1</td>
<td>29.7</td>
</tr>
<tr>
<td>February, 2</td>
<td>33.4</td>
</tr>
<tr>
<td>March, 3</td>
<td>39.0</td>
</tr>
<tr>
<td>April, 4</td>
<td>48.2</td>
</tr>
<tr>
<td>May, 5</td>
<td>57.2</td>
</tr>
<tr>
<td>June, 6</td>
<td>66.9</td>
</tr>
<tr>
<td>July, 7</td>
<td>73.5</td>
</tr>
<tr>
<td>August, 8</td>
<td>71.4</td>
</tr>
<tr>
<td>September, 9</td>
<td>62.3</td>
</tr>
<tr>
<td>October, 10</td>
<td>51.4</td>
</tr>
<tr>
<td>November, 11</td>
<td>39.0</td>
</tr>
<tr>
<td>December, 12</td>
<td>31.0</td>
</tr>
</tbody>
</table>

**SOURCE:** U.S. National Oceanic and Atmospheric Administration

Notice that the scatter diagram looks like the graph of a sinusoidal function. We choose to fit the data to a sine function of the form

\[ y = A \sin(\omega x - \phi) + B \]

where \( A, B, \omega, \) and \( \phi \) are constants.

### Example 3

#### Finding a Sinusoidal Function from Temperature Data

Fit a sine function to the data in Table 12.

**Solution**

We begin with a scatter diagram of the data for one year. See Figure 108. The data will be fitted to a sine function of the form

\[ y = A \sin(\omega x - \phi) + B \]

**STEP 1:** To find the amplitude \( A \), we compute

\[
\text{Amplitude} = \frac{\text{largest data value} - \text{smallest data value}}{2} = \frac{73.5 - 29.7}{2} = 21.9
\]

To see the remaining steps in this process, we superimpose the graph of the function \( y = 21.9 \sin x \), where \( x \) represents months, on the scatter diagram.
Figure 109 shows the two graphs. To fit the data, the graph needs to be shifted vertically, shifted horizontally, and stretched horizontally.

**STEP 2:** We determine the vertical shift by finding the average of the highest and lowest data values.

\[
\text{Vertical shift} = \frac{73.5 + 29.7}{2} = 51.6
\]

Now we superimpose the graph of \( y = 21.9 \sin x + 51.6 \) on the scatter diagram. See Figure 110.

We see that the graph needs to be shifted horizontally and stretched horizontally.

**STEP 3:** It is easier to find the horizontal stretch factor first. Since the temperatures repeat every 12 months, the period of the function is \( T = 12 \). Since

\[
T = \frac{2\pi}{\omega} = 12
\]

we find

\[
\omega = \frac{2\pi}{12} = \frac{\pi}{6}
\]

Now we superimpose the graph of \( y = 21.9 \sin \left( \frac{\pi}{6} x \right) + 51.6 \) on the scatter diagram. See Figure 111. We see that the graph still needs to be shifted horizontally.

**STEP 4:** To determine the horizontal shift, we use the period \( T = 12 \) and divide the interval \([0, 12]\) into four subintervals of length \( T/4 = 3 \):

\[
[0, 3], \ [3, 6], \ [6, 9], \ [9, 12]
\]

The sine curve is increasing on the interval \((0, 3)\) and is decreasing on the interval \((3, 9)\), so a local maximum occurs at \( x = 3 \). The data indicate that a maximum occurs at \( x = 7 \) (corresponding to July’s temperature), so we must shift the graph of the function 4 units to the right by replacing \( x \) by \( x - 4 \). Doing this, we obtain

\[
y = 21.9 \sin \left( \frac{\pi}{6} (x - 4) \right) + 51.6
\]

Multiplying out, we find that a sine function of the form \( y = A \sin(\omega x - \phi) + B \) that fits the data is

\[
y = 21.9 \sin \left( \frac{\pi}{6} x - \frac{2\pi}{3} \right) + 51.6
\]

The graph of \( y = 21.9 \sin \left( \frac{\pi}{6} x - \frac{2\pi}{3} \right) + 51.6 \) and the scatter diagram of the data are shown in Figure 112.
The steps to fit a sine function
\[ y = A \sin(\omega x - \phi) + B \]
to sinusoidal data follow:

**Steps for Fitting Data to a Sine Function**

1. **Step 1:** Determine \( A \), the amplitude of the function.
   \[ \text{Amplitude} = \frac{\text{largest data value} - \text{smallest data value}}{2} \]
2. **Step 2:** Determine \( B \), the vertical shift of the function.
   \[ \text{Vertical shift} = \frac{\text{largest data value} + \text{smallest data value}}{2} \]
3. **Step 3:** Determine \( \omega \). Since the period \( T \), the time it takes for the data to repeat, is \( T = \frac{2\pi}{\omega} \), we have
   \[ \omega = \frac{2\pi}{T} \]
4. **Step 4:** Determine the horizontal shift of the function by using the period of the data. Divide the period into four subintervals of equal length. Determine the \( x \)-coordinate for the maximum of the sine function and the \( x \)-coordinate for the maximum value of the data. Use this information to determine the value of the phase shift, \( \frac{\phi}{\omega} \).

**Now Work Problem 29(a)–(c)**

Let’s look at another example. Since the number of hours of sunlight in a day cycles annually, the number of hours of sunlight in a day for a given location can be modeled by a sinusoidal function.

The longest day of the year (in terms of hours of sunlight) occurs on the day of the summer solstice. For locations in the northern hemisphere, the summer solstice is the time when the sun is farthest north. In 2005, the summer solstice occurred on June 21 (the 172nd day of the year) at 2:46 AM EDT. The shortest day of the year occurs on the day of the winter solstice. The winter solstice is the time when the Sun is farthest south (again, for locations in the northern hemisphere). In 2005, the winter solstice occurred on December 21 (the 355th day of the year) at 1:35 PM (EST).

**Example 4 Finding a Sinusoidal Function for Hours of Daylight**

According to the *Old Farmer’s Almanac*, the number of hours of sunlight in Boston on the summer solstice is 15.30 and the number of hours of sunlight on the winter solstice is 9.08.

(a) Find a sinusoidal function of the form \( y = A \sin(\omega x - \phi) + B \) that fits the data.

(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.

(c) Draw a graph of the function found in part (a).

(d) Look up the number of hours of sunlight for April 1 in the *Old Farmer’s Almanac* and compare it to the results found in part (b).

**Source:** The *Old Farmer’s Almanac*, www.almanac.com/rise

**Solution**

(a) **Step 1:** Amplitude
\[ \text{Amplitude} = \frac{\text{largest data value} - \text{smallest data value}}{2} \]
\[ = \frac{15.30 - 9.08}{2} = 3.11 \]
STEP 2: Vertical shift $= \frac{\text{largest data value} + \text{smallest data value}}{2}$

$= \frac{15.30 + 9.08}{2} = 12.19$

STEP 3: The data repeat every 365 days. Since $T = \frac{2\pi}{\omega} = 365$, we find $\omega = \frac{2\pi}{365}$

So far, we have $y = 3.11 \sin \left( \frac{2\pi}{365} x - \phi \right) + 12.19$.

STEP 4: To determine the horizontal shift, we use the period $T = 365$ and divide the interval $[0, 365]$ into four subintervals of length $365 \div 4 = 91.25$:

$[0, 91.25], \ [91.25, 182.5], \ [182.5, 273.75], \ [273.75, 365]$  

The sine curve is increasing on the interval $(0, 91.25)$ and decreasing on the interval $(91.25, 273.75)$, so a local maximum occurs at $x = 91.25$. Since the maximum occurs on the summer solstice at $x = 172$, we must shift the graph of the function $172 - 91.25 = 80.75$ units to the right by replacing $x$ by $x - 80.75$. Doing this, we obtain

$y = 3.11 \sin \left( \frac{2\pi}{365} (x - 80.75) \right) + 12.19$

Multiplying out, we find that a sine function of the form $y = A \sin(\omega x - \phi) + B$ that fits the data is

$y = 3.11 \sin \left( \frac{2\pi}{365} x - \frac{323\pi}{730} \right) + 12.19$

(b) To predict the number of hours of daylight on April 1, we let $x = 91$ in the function found in part (a) and obtain

$y = 3.11 \sin \left( \frac{2\pi}{365} \cdot 91 - \frac{323\pi}{730} \right) + 12.19$

$\approx 12.74$

So we predict that there will be about 12.74 hours = 12 hours, 44 minutes of sunlight on April 1 in Boston.

(c) The graph of the function found in part (a) is given in Figure 113.

(d) According to the *Old Farmer’s Almanac*, there will be 12 hours 45 minutes of sunlight on April 1 in Boston.

**Now Work Problem 35**

Certain graphing utilities (such as a TI-83, TI-84 Plus, and TI-86) have the capability of finding the sine function of best fit for sinusoidal data. At least four data points are required for this process.

**Example 5** Finding the Sine Function of Best Fit

Use a graphing utility to find the sine function of best fit for the data in Table 12. Graph this function with the scatter diagram of the data.

**Solution** Enter the data from Table 12 and execute the SINe REGression program. The result is shown in Figure 114.
The output that the utility provides shows the equation

\[ y = a \sin(bx + c) + d \]

The sinusoidal function of best fit is

\[ y = 21.15 \sin(0.55x - 2.35) + 51.19 \]

where \( x \) represents the month and \( y \) represents the average temperature.

Figure 115 shows the graph of the sinusoidal function of best fit on the scatter diagram.

Now Work

PROBLEMS 29(d) AND (e)

7.8 Assess Your Understanding

Concepts and Vocabulary

1. For the graph of \( y = A \sin(\omega x - \phi) \), the number \( \frac{\phi}{\omega} \) is called the _____.

2. True or False Only two data points are required by a graphing utility to find the sine function of best fit.

Skill Building

In Problems 3–14, find the amplitude, period, and phase shift of each function. Graph each function. Be sure to label key points. Show at least two periods.

3. \( y = 4 \sin(2x - \pi) \)

4. \( y = 3 \sin(3x - \pi) \)

5. \( y = 2 \cos\left(3x + \frac{\pi}{2}\right)\)

6. \( y = 3 \cos(2x + \pi) \)

7. \( y = -3 \sin\left(2x + \frac{\pi}{2}\right)\)

8. \( y = -2 \cos\left(2x - \pi \right)\)

9. \( y = 4 \sin(\pi x + 2) - 5 \)

10. \( y = 2 \cos(2\pi x + 4) + 4 \)

11. \( y = 3 \cos(\pi x - 2) + 5 \)

12. \( y = 2 \cos(2\pi x - 4) - 1 \)

13. \( y = -3 \sin\left(-2x + \frac{\pi}{2}\right)\)

14. \( y = -3 \cos\left(-2x + \frac{\pi}{2}\right)\)

In Problems 15–18, write the equation of a sine function that has the given characteristics.

15. Amplitude: 2
   Period: \( \pi \)
   Phase shift: \( \frac{1}{2} \)

16. Amplitude: 3
    Period: \( \frac{\pi}{2} \)
    Phase shift: 2

17. Amplitude: 3
    Period: \( 3\pi \)
    Phase shift: \(-\frac{1}{3}\)

18. Amplitude: 2
    Period: \( \pi \)
    Phase shift: -2

Applications and Extensions

In Problems 19–26, apply the methods of this and the previous section to graph each function. Be sure to label key points and show at least two periods.

19. \( y = 2 \tan(4x - \pi) \)

20. \( y = \frac{1}{2} \cot(2x - \pi) \)

21. \( y = 3 \csc\left(2x - \frac{\pi}{4}\right)\)

22. \( y = \frac{1}{2} \sec(3x - \pi) \)

23. \( y = -\cot\left(2x + \frac{\pi}{2}\right)\)

24. \( y = -\tan\left(3x + \frac{\pi}{2}\right)\)

25. \( y = -\sec(2\pi x + \pi) \)

26. \( y = -\csc\left(-\frac{1}{2} \pi x + \frac{\pi}{4}\right)\)
27. Alternating Current (ac) Circuits The current $I$, in amperes, flowing through an ac (alternating current) circuit at time $t$, in seconds, is

$$I(t) = 120 \sin \left(30\pi t - \frac{\pi}{3}\right), \quad t \geq 0$$

What is the period? What is the amplitude? What is the phase shift? Graph this function over two periods.

28. Alternating Current (ac) Circuits The current $I$, in amperes, flowing through an ac (alternating current) circuit at time $t$, in seconds, is

$$I(t) = 220 \sin \left(60\pi t - \frac{\pi}{6}\right), \quad t \geq 0$$

What is the period? What is the amplitude? What is the phase shift? Graph this function over two periods.

29. Monthly Temperature The following data represent the average monthly temperatures for Juneau, Alaska.

<table>
<thead>
<tr>
<th>Month, x</th>
<th>Average Monthly Temperature, °F</th>
</tr>
</thead>
<tbody>
<tr>
<td>January, 1</td>
<td>24.2</td>
</tr>
<tr>
<td>February, 2</td>
<td>28.4</td>
</tr>
<tr>
<td>March, 3</td>
<td>32.7</td>
</tr>
<tr>
<td>April, 4</td>
<td>39.7</td>
</tr>
<tr>
<td>May, 5</td>
<td>47.0</td>
</tr>
<tr>
<td>June, 6</td>
<td>53.0</td>
</tr>
<tr>
<td>July, 7</td>
<td>56.0</td>
</tr>
<tr>
<td>August, 8</td>
<td>55.0</td>
</tr>
<tr>
<td>September, 9</td>
<td>49.4</td>
</tr>
<tr>
<td>October, 10</td>
<td>42.2</td>
</tr>
<tr>
<td>November, 11</td>
<td>32.0</td>
</tr>
<tr>
<td>December, 12</td>
<td>27.1</td>
</tr>
</tbody>
</table>

Source: U.S. National Oceanic and Atmospheric Administration

(a) Draw a scatter diagram of the data for one period.
(b) Find a sinusoidal function of the form
$$y = A \sin(\omega x - \phi) + B$$
that fits the data.
(c) Draw the sinusoidal function found in part (b) on the scatter diagram.

(d) Use a graphing utility to find the sinusoidal function of best fit.
(e) Draw the sinusoidal function of best fit on a scatter diagram of the data.

30. Monthly Temperature The following data represent the average monthly temperatures for Washington, D.C.

(a) Draw a scatter diagram of the data for one period.
(b) Find a sinusoidal function of the form
$$y = A \sin(\omega x - \phi) + B$$
that fits the data.
(c) Draw the sinusoidal function found in part (b) on the scatter diagram.

(d) Use a graphing utility to find the sinusoidal function of best fit.
(e) Draw the sinusoidal function of best fit on a scatter diagram of the data.

31. Monthly Temperature The following data represent the average monthly temperatures for Indianapolis, Indiana.

(a) Draw a scatter diagram of the data for one period.
(b) Find a sinusoidal function of the form
$$y = A \sin(\omega x - \phi) + B$$
that fits the data.
(c) Draw the sinusoidal function found in part (b) on the scatter diagram.

(d) Use a graphing utility to find the sinusoidal function of best fit.
(e) Graph the sinusoidal function of best fit on a scatter diagram of the data.

32. Monthly Temperature The following data represent the average monthly temperatures for Baltimore, Maryland.

(a) Draw a scatter diagram of the data for one period.
33. Tides  Suppose that the length of time between consecutive high tides is approximately 12.5 hours. According to the National Oceanic and Atmospheric Administration, on Saturday, August 7, 2004, in Savannah, Georgia, high tide occurred at 3:38 AM (3.6333 hours) and low tide occurred at 10:08 AM (10.1333 hours). Water heights are measured as the amounts above or below the mean lower low water. The height of the water at high tide was 13.2 feet, and the height of the water at low tide was 0.6 foot.

(a) Approximately when will the next high tide occur?
(b) Find a sinusoidal function of the form
\[ y = A \sin(\omega x - \phi) + B \]
that fits the data.
(c) Draw the sinusoidal function found in part (b) on the scatter diagram.
(d) Use a graphing utility to find the sinusoidal function of best fit.
(c) Graph the sinusoidal function of best fit on a scatter diagram of the data.

34. Tides  Suppose that the length of time between consecutive high tides is approximately 12.5 hours. According to the National Oceanic and Atmospheric Administration, on Saturday, August 7, 2004, in Juneau, Alaska, high tide occurred at 8:11 AM (8.1833 hours) and low tide occurred at 2:14 PM (14.2333 hours). Water heights are measured as the amounts above or below the mean lower low water. The height of the water at high tide was 8.2 feet, and the height of the water at low tide was 2.2 feet.

(a) Approximately when will the next high tide occur?
(b) Find a sinusoidal function of the form
\[ y = A \sin(\omega x - \phi) + B \]
that fits the data.

35. Hours of Daylight  According to the Old Farmer's Almanac, in Anchorage, Alaska, the number of hours of sunlight on the summer solstice of 2005 is 19.42 and the number of hours of sunlight on the winter solstice is 5.47.

(a) Find a sinusoidal function of the form
\[ y = A \sin(\omega x - \phi) + B \]
that fits the data.
(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.
(c) Draw a graph of the function found in part (a).
(d) Look up the number of hours of sunlight for April 1 in the Old Farmer's Almanac, and compare the actual hours of daylight to the results found in part (c).

36. Hours of Daylight  According to the Old Farmer's Almanac, in Detroit, Michigan, the number of hours of sunlight on the summer solstice of 2005 is 15.30 and the number of hours of sunlight on the winter solstice is 9.08.

(a) Find a sinusoidal function of the form
\[ y = A \sin(\omega x - \phi) + B \]
that fits the data.
(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.
(c) Draw a graph of the function found in part (a).
(d) Look up the number of hours of sunlight for April 1 in the Old Farmer's Almanac, and compare the actual hours of daylight to the results found in part (c).

37. Hours of Daylight  According to the Old Farmer's Almanac, in Miami, Florida, the number of hours of sunlight on the summer solstice of 2005 is 13.75 and the number of hours of sunlight on the winter solstice is 10.53.

(a) Find a sinusoidal function of the form
\[ y = A \sin(\omega x - \phi) + B \]
that fits the data.
(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.
(c) Draw a graph of the function found in part (a).
(d) Look up the number of hours of sunlight for April 1 in the Old Farmer's Almanac, and compare the actual hours of daylight to the results found in part (c).

Discussion and Writing

39. Explain how the amplitude and period of a sinusoidal graph are used to establish the scale on each coordinate axis.

40. Find an application in your major field that leads to a sinusoidal graph. Write a paper about your findings.
CHAPTER REVIEW

Things to Know

Definitions

- **Angle in standard position** (p. 504)
  Vertex is at the origin; initial side is along the positive x-axis.

- **1 Degree (1°)** (p. 505)
  $1^\circ = \frac{1}{360}$ revolution

- **1 Radian** (p. 507)
  The measure of a central angle of a circle whose rays subtend an arc whose length is the radius of the circle

- **Acute angle** (p. 518)
  An angle whose measure is $0^\circ < \theta < 90^\circ$ or $0 < \theta < \frac{\pi}{2}$, $\theta$ in radians

- **Complementary angles** (p. 523)
  Two acute angles whose sum is $90^\circ$ or $\frac{\pi}{2}$ radians

- **Cofunction** (p. 524)
  The following pairs of functions are cofunctions of each other: sine and cosine; tangent and cotangent; secant and cosecant.

- **Trigonometric functions of a general angle** (p. 540)
  $P = (a, b)$ is the point on the terminal side of $\theta$ a distance $r$ from the origin:

  - $\sin \theta = \frac{b}{r}$
  - $\cos \theta = \frac{a}{r}$
  - $\tan \theta = \frac{b}{a}$, $a \neq 0$

  - $\csc \theta = \frac{r}{b}$, $b \neq 0$
  - $\sec \theta = \frac{r}{a}$, $a \neq 0$
  - $\cot \theta = \frac{a}{b}$, $b \neq 0$

- **Reference angle of $\theta$** (p. 545)
  The acute angle formed by the terminal side of $\theta$ and either the positive or negative x-axis

- **Periodic function** (p. 555)
  $f(\theta + p) = f(\theta)$, for all $\theta$, $p > 0$, where the smallest such $p$ is the fundamental period

Formulas

- **1 revolution** = $360^\circ$ (p. 505)
  $= 2\pi$ radians (p. 509)
  $s = r\theta$ (p. 508)

- **$A = \frac{1}{2}r^2\theta$** (p. 511)

- **$v = rw$** (p. 512)

  $\theta$ is measured in radians; $s$ is the length of the arc subtended by the central angle $\theta$ of the circle of radius $r$;

  $A$ is the area of the sector of a circle of radius $r$ formed by a central angle of $\theta$ radians.

  $v$ is the linear speed along the circle of radius $r$; $\omega$ is the angular speed (measured in radians per unit time).

Table of Values

<table>
<thead>
<tr>
<th>$\theta$ (Radians)</th>
<th>$\theta$ (Degrees)</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\tan \theta$</th>
<th>$\csc \theta$</th>
<th>$\sec \theta$</th>
<th>$\cot \theta$</th>
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<tbody>
<tr>
<td>0</td>
<td>0°</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>1</td>
<td>Not defined</td>
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<td>$\frac{\sqrt{3}}{3}$</td>
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<td>$\frac{\pi}{2}$</td>
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<td>Not defined</td>
<td>1</td>
<td>Not defined</td>
<td>0</td>
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<tr>
<td>$\pi$</td>
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<td>0</td>
<td>Not defined</td>
<td>$-1$</td>
<td>Not defined</td>
<td>0</td>
</tr>
</tbody>
</table>
Fundamental Identities (p. 520)
\[
\begin{align*}
\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta} \\
csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\
\sin^2 \theta + \cos^2 \theta &= 1 & \tan^2 \theta + 1 &= \sec^2 \theta & \cot^2 \theta + 1 &= \csc^2 \theta
\end{align*}
\]

Properties of the Trigonometric Functions

**y = sin x** (p. 561)
- Domain: \(-\infty < x < \infty\)
- Range: \(-1 \leq y \leq 1\)
- Periodic: period = \(2\pi\) (360°)
- Odd function

**y = cos x** (pp. 562–563)
- Domain: \(-\infty < x < \infty\)
- Range: \(-1 \leq y \leq 1\)
- Periodic: period = \(2\pi\) (360°)
- Even function

**y = tan x** (pp. 575–576)
- Domain: \(-\infty < x < \infty\), except odd integer multiples of \(\frac{\pi}{2}\) (90°)
- Range: \(-\infty < y < \infty\)
- Periodic: period = \(\pi\) (180°)
- Odd function
- Vertical asymptotes at odd integer multiples of \(\frac{\pi}{2}\)

**y = cot x** (pp. 577)
- Domain: \(-\infty < x < \infty\), except integer multiples of \(\pi\) (180°)
- Range: \(-\infty < y < \infty\)
- Periodic: period = \(\pi\) (180°)
- Odd function
- Vertical asymptotes at integer multiples of \(\pi\)

**y = csc x** (p. 578)
- Domain: \(-\infty < x < \infty\), except integer multiples of \(\pi\) (180°)
- Range: \(|y| \geq 1\)
- Periodic: period = \(2\pi\) (360°)
- Odd function
- Vertical asymptotes at integer multiples of \(\pi\)

**y = sec x** (p. 578)
- Domain: \(-\infty < x < \infty\), except odd integer multiples of \(\frac{\pi}{2}\) (90°)
- Range: \(|y| \geq 1\)
- Periodic: period = \(2\pi\) (360°)
- Even function
- Vertical asymptotes at odd integer multiples of \(\frac{\pi}{2}\)

**Sinusoidal Graphs**

\[
\begin{align*}
y &= A \sin(\omega x) + B, \quad \omega > 0 & \text{Period} &= \frac{2\pi}{\omega} \ (\text{pp. 565, 582)} \\
y &= A \cos(\omega x) + B, \quad \omega > 0 & \text{Amplitude} &= |A| \ (\text{pp. 565, 582)} \\
y &= A \sin(\omega x - \phi) + B = A \sin\left[\omega \left(x - \frac{\phi}{\omega}\right)\right] + B & \text{Phase shift} &= \frac{\phi}{\omega} \ (\text{p. 582)} \\
y &= A \cos(\omega x - \phi) + B = A \cos\left[\omega \left(x - \frac{\phi}{\omega}\right)\right] + B
\end{align*}
\]
## Objectives

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<td>3 Determine the signs of the trigonometric functions of an angle in a given quadrant (p. 544)</td>
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<tr>
<td></td>
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<td></td>
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<td>100, 101</td>
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## Review Exercises

In Problems 1–4, convert each angle in degrees to radians. Express your answer as a multiple of \( \pi \).

1. \( 135^\circ \)  
2. \( 210^\circ \)  
3. \( 18^\circ \)  
4. \( 15^\circ \)

In Problems 5–8, convert each angle in radians to degrees.

5. \( \frac{3\pi}{4} \)  
6. \( \frac{2\pi}{3} \)  
7. \( -\frac{5\pi}{2} \)  
8. \( -\frac{3\pi}{2} \)
Review Exercises 595

In Problems 9–30, find the exact value of each expression. Do not use a calculator.

9. \( \tan \frac{\pi}{4} - \sin \frac{\pi}{6} \)
10. \( \cos \frac{\pi}{3} + \sin \frac{\pi}{2} \)
11. \( 3 \sin 45^\circ - 4 \tan \frac{\pi}{6} \)
12. \( 4 \cos 60^\circ + 3 \tan \frac{\pi}{3} \)
13. \( 6 \cos \frac{3\pi}{4} + 2 \tan \left(\frac{\pi}{3}\right) \)
14. \( 3 \sin \frac{2\pi}{3} - 4 \cos \frac{5\pi}{2} \)
15. \( \sec \left(\frac{-\pi}{3}\right) - \cot \left(\frac{-5\pi}{4}\right) \)
16. \( 4 \csc \frac{3\pi}{4} - \cot \left(\frac{-\pi}{4}\right) \)
17. \( \tan \pi + \sin \pi \)
18. \( \cos \frac{\pi}{2} - \csc \left(\frac{-\pi}{2}\right) \)
19. \( \cos 540^\circ - \tan(-405^\circ) \)
20. \( \sin 270^\circ + \cos(-180^\circ) \)
21. \( \sin^2 20^\circ + \frac{1}{\sec^2 20^\circ} \)
22. \( \frac{1}{\cos^2 40^\circ} - \frac{1}{\cot^2 40^\circ} \)
23. \( \sec 50^\circ \cos 50^\circ \)
24. \( \tan 10^\circ \cot 10^\circ \)
25. \( \frac{\sin 50^\circ}{\cos 40^\circ} \)
26. \( \frac{\tan 20^\circ}{\cot 70^\circ} \)
27. \( \frac{\sin(-40^\circ)}{\cos 50^\circ} \)
28. \( \tan(-20^\circ) \cot 20^\circ \)
29. \( \sin 400^\circ \sec(-50^\circ) \)
30. \( \cot 200^\circ \cot(-70^\circ) \)

In Problems 31–46, find the exact value of each of the remaining trigonometric functions.

31. \( \sin \theta = \frac{4}{5}, \) \( \theta \) is acute
32. \( \tan \theta = \frac{1}{4}, \) \( \theta \) is acute
33. \( \tan \theta = \frac{12}{5}, \) \( \sin \theta < 0 \)
34. \( \cot \theta = \frac{12}{5}, \) \( \cos \theta < 0 \)
35. \( \sec \theta = -\frac{5}{4}, \) \( \tan \theta < 0 \)
36. \( \csc \theta = -\frac{5}{3}, \) \( \cot \theta < 0 \)
37. \( \sin \theta = \frac{12}{13}, \) \( \theta \) in quadrant II
38. \( \cos \theta = -\frac{3}{5}, \) \( \theta \) in quadrant III
39. \( \sin \theta = -\frac{5}{13}, \) \( \frac{3\pi}{2} < \theta < 2\pi \)
40. \( \cos \theta = \frac{12}{13}, \) \( \frac{3\pi}{2} < \theta < 2\pi \)
41. \( \tan \theta = \frac{1}{3}, \) \( 180^\circ < \theta < 270^\circ \)
42. \( \tan \theta = -\frac{2}{3}, \) \( 90^\circ < \theta < 180^\circ \)
43. \( \sec \theta = 3, \) \( \frac{3\pi}{2} < \theta < 2\pi \)
44. \( \csc \theta = -4, \) \( \pi < \theta < \frac{3\pi}{2} \)
45. \( \cot \theta = -2, \) \( \frac{\pi}{2} < \theta < \pi \)
46. \( \tan \theta = -2, \) \( \frac{3\pi}{2} < \theta < 2\pi \)

In Problems 47–62, graph each function. Each graph should contain at least two periods.

47. \( y = 2 \sin(4x) \)
48. \( y = -3 \cos(2x) \)
49. \( y = -2 \cos\left(x + \frac{\pi}{2}\right) \)
50. \( y = 3 \sin(x - \pi) \)
51. \( y = \tan(x + \pi) \)
52. \( y = -\tan\left(x - \frac{\pi}{2}\right) \)
53. \( y = -2 \tan(3x) \)
54. \( y = 4 \tan(2x) \)
55. \( y = \cot\left(x + \frac{\pi}{4}\right) \)
56. \( y = -4 \cot(2x) \)
57. \( y = 4 \sec(2x) \)
58. \( y = \csc\left(x + \frac{\pi}{4}\right) \)
59. \( y = 4 \sin(2x + 4) - 2 \)
60. \( y = 3 \cos(4x + 2) + 1 \)
61. \( y = 4 \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \)
62. \( y = 5 \cot\left(\frac{x}{3} - \frac{\pi}{4}\right) \)

In Problems 63–66, determine the amplitude and period of each function without graphing.

63. \( y = 4 \cos x \)
64. \( y = \sin(2x) \)
65. \( y = -8 \sin\left(\frac{\pi}{2} x\right) \)
66. \( y = -2 \cos(3\pi x) \)

In Problems 67–74, find the amplitude, period, and phase shift of each function. Graph each function. Show at least two periods.

67. \( y = 4 \sin(3x) \)
68. \( y = 2 \cos\left(\frac{1}{2} x\right) \)
69. \( y = 2 \sin(2x - \pi) \)
70. \( y = -\cos\left(\frac{1}{2} x + \frac{\pi}{2}\right) \)
71. \( y = \frac{1}{2} \sin\left(\frac{3}{2} x - \pi\right) \)
72. \( y = \frac{3}{2} \cos(6x + 3\pi) \)
73. \( y = -\frac{2}{5} \cos(\pi x - 6) \)
74. \( y = -7 \sin\left(\frac{\pi}{3} x + \frac{4}{3}\right) \)
In Problems 75–78, find a function whose graph is given.

75.  

76.  

77.  

78.  

79. Find the value of each of the six trigonometric functions of the angle \( \theta \) in the illustration.

80. Use a calculator to approximate \( \sec 10^\circ \). Round the answer to two decimal places.

81. Find the exact value of each of the six trigonometric functions of an angle \( \theta \) if \((3, -4)\) is a point on the terminal side of \( \theta \).

82. Name the quadrant \( \theta \) lies in if \( \cos \theta > 0 \) and \( \tan \theta < 0 \).

83. Find the reference angle of \(-\frac{4\pi}{5}\).

84. Find the exact value of \( \sin t \), \( \cos t \), and \( \tan t \) if \( P = \left(\frac{3}{5}, \frac{4}{5}\right) \) is the point on the unit circle that corresponds to \( t \).

85. What is the domain and the range of the secant function?

86. (a) Convert the angle \( 32^\circ 20' 35'' \) to a decimal in degrees. Round the answer to two decimal places.

(b) Convert the angle \( 63.18^\circ \) to \( D'M'S'' \) form. Express the answer to the nearest second.

87. Find the length of the arc subtended by a central angle of \( 30^\circ \) on a circle of radius 2 feet. What is the area of the sector?

88. The minute hand of a clock is 8 inches long. How far does the tip of the minute hand move in 30 minutes? How far does it move in 20 minutes?

89. Angular Speed of a Race Car A race car is driven around a circular track at a constant speed of 180 miles per hour. If the diameter of the track is \( \frac{1}{2} \) mile, what is the angular speed of the car? Express your answer in revolutions per hour (which is equivalent to laps per hour).

90. Merry-Go-Rounds A neighborhood carnival has a merry-go-round whose radius is 25 feet. If the time for one revolution is 30 seconds, how fast is the merry-go-round going? Give the linear speed and the angular speed.

91. Lighthouse Beacons The Montauk Point Lighthouse on Long Island has dual beams (two light sources opposite each other). Ships at sea observe a blinking light every 5 seconds. What rotation speed is required to do this?

92. Spin Balancing Tires The radius of each wheel of a car is 16 inches. At how many revolutions per minute should a spin balancer be set to balance the tires at a speed of 90 miles per hour? Is the setting different for a wheel of radius 14 inches? If so, what is this setting?

93. Measuring the Length of a Lake From a stationary hot-air balloon 500 feet above the ground, two sightings of a lake are made (see the figure). How long is the lake?

94. Finding the Speed of a Glider From a glider 200 feet above the ground, two sightings of a stationary object directly in front are taken 1 minute apart (see the figure). What is the speed of the glider?
95. Finding the Width of a River  
Find the distance from A to C across the river illustrated in the figure.

96. Finding the Height of a Building  
Find the height of the building shown in the figure.

97. Finding the Distance to Shore  
The Sears Tower in Chicago is 1454 feet tall and is situated about 1 mile inland from the shore of Lake Michigan, as indicated in the figure. An observer in a pleasure boat on the lake directly in front of the Sears Tower looks at the top of the tower and measures the angle of elevation as $5^\circ$. How far offshore is the boat?

98. Alternating Voltage  
The electromotive force $E$, in volts, in a certain ac circuit obeys the equation

$$E = 120 \sin(120\pi t), \quad t \geq 0$$

where $t$ is measured in seconds.

(a) What is the maximum value of $E$?
(b) What is the period?
(c) Graph this function over two periods.

99. Alternating Current  
The current $I$, in amperes, flowing through an ac (alternating current) circuit at time $t$ is

$$I = 220 \sin\left(30\pi t + \frac{\pi}{6}\right), \quad t \geq 0$$

(a) What is the period?
(b) What is the amplitude?
(c) What is the phase shift?
(d) Graph this function over two periods.

100. Monthly Temperature  
The following data represent the average monthly temperatures for Phoenix, Arizona.

<table>
<thead>
<tr>
<th>Month, $m$</th>
<th>Average Monthly Temperature, $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>January, 1</td>
<td>51</td>
</tr>
<tr>
<td>February, 2</td>
<td>55</td>
</tr>
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<td>March, 3</td>
<td>63</td>
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<td>April, 4</td>
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<td>May, 5</td>
<td>77</td>
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<tr>
<td>June, 6</td>
<td>86</td>
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<tr>
<td>July, 7</td>
<td>90</td>
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<tr>
<td>August, 8</td>
<td>90</td>
</tr>
<tr>
<td>September, 9</td>
<td>84</td>
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<td>October, 10</td>
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<td>November, 11</td>
<td>59</td>
</tr>
<tr>
<td>December, 12</td>
<td>52</td>
</tr>
</tbody>
</table>

Source: U.S. National Oceanic and Atmospheric Administration

(a) Draw a scatter diagram of the data for one period.
(b) Find a sinusoidal function of the form $y = A \sin(\omega x - \phi) + B$ that fits the data.
(c) Draw the sinusoidal function found in part (b) on the scatter diagram.
(d) Use a graphing utility to find the sinusoidal function of best fit.
(e) Graph the sinusoidal function of best fit on the scatter diagram.

101. Hours of Daylight  
According to the *Old Farmer’s Almanac*, in Las Vegas, Nevada, the number of hours of sunlight on the summer solstice is 14.63 and the number of hours of sunlight on the winter solstice is 9.72.

(a) Find a sinusoidal function of the form $y = A \sin(\omega x - \phi) + B$ that fits the data.
(b) Use the function found in part (a) to predict the number of hours of sunlight on April 1, the 91st day of the year.
(c) Draw a graph of the function found in part (a).
(d) Look up the number of hours of sunlight for April 1 in the *Old Farmer’s Almanac* and compare the actual hours of daylight to the results found in part (c).
CHAPTER TEST

In Problems 1–3, convert each angle in degrees to radians. Express your answer as a multiple of $\pi$.
1. $260^\circ$  
2. $-400^\circ$  
3. $13^\circ$

In Problems 4–6 convert each angle in radians to degrees.
4. $-\frac{\pi}{8}$  
5. $\frac{9\pi}{2}$  
6. $\frac{3\pi}{4}$

In Problems 7–12, find the exact value of each expression.
7. $\sin \frac{\pi}{6}$  
8. $\cos \left( -\frac{5\pi}{4} \right) - \cos \frac{3\pi}{4}$  
9. $\cos(-120^\circ)$  
10. $\tan 330^\circ$  
11. $\sin \frac{\pi}{2} - \tan \frac{19\pi}{4}$  
12. $2 \sin^2 60^\circ - 3 \cos 45^\circ$

In Problems 13–16, use a calculator to evaluate each expression. Round your answer to three decimal places.
13. $\sin 17^\circ$  
14. $\cos \frac{2\pi}{5}$  
15. $\sec 229^\circ$  
16. $\cot \frac{28\pi}{9}$

17. Fill in each table entry with the sign of each function.

<table>
<thead>
<tr>
<th>$\theta$ in QI</th>
<th>$\cos \theta$</th>
<th>$\tan \theta$</th>
<th>$\sec \theta$</th>
<th>$\csc \theta$</th>
<th>$\cot \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$ in QII</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta$ in QIII</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta$ in QIV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

18. If $f(x) = \sin x$ and $f(a) = \frac{3}{5}$, find $f(-a)$.

In Problems 19–21 find the value of the remaining five trigonometric functions of $\theta$.
19. $\sin \theta = \frac{5}{7}$, $\theta$ in quadrant II  
20. $\cos \theta = \frac{2}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$  
21. $\tan \theta = -\frac{12}{5}$, $\frac{\pi}{2} < \theta < \pi$

In Problems 22–24, the point $(x, y)$ is on the terminal side of angle $\theta$ in standard position. Find the exact value of the given trigonometric function.
22. $(2, 7)$, $\sin \theta$  
23. $(-5, 11)$, $\cos \theta$  
24. $(6, -3)$, $\tan \theta$

In Problems 25 and 26, graph the function.
25. $y = 2 \sin \left( x - \frac{\pi}{6} \right)$  
26. $y = \tan \left( -x + \frac{\pi}{4} \right) + 2$

27. Write an equation for a sinusoidal graph with the following properties:
   $A = -3$  
   period $= \frac{2\pi}{3}$  
   phase shift $= -\frac{\pi}{4}$

28. Logan has a garden in the shape of a sector of a circle; the outer rim of the garden is 25 feet long and the central angle of the sector is 50°. She wants to add a 3 foot-wide walk to the outer rim; how many square feet of paving blocks will she need to build the walk?

29. Hungarian Adrian Annus won the gold medal for the hammer throw at the 2004 Olympics in Athens with a winning distance of 83.19 meters. * The event consists of swinging a 16 pound weight attached to a wire 190 centimeters long—in a circle and then releasing it. Assuming his release is at a 45° angle to the ground, the hammer will travel a distance of $\frac{v_0}{g}$ meters, where $g = 9.8$ meters/second² and $v_0$ is the linear speed of the hammer when released. At what rate (rpm) was he swinging the hammer upon release?

30. A ship is just offshore of New York City. A sighting is taken of the Statue of Liberty, which is about 305 feet tall. If the angle of elevation to the top of the statue is 20°, how far is the ship from the base of the statue?

31. To measure the height of a building, two sightings are taken a distance of 50 feet apart. If the first angle of elevation is 45° and the second is 32°, what is the height of the building?

* Annus was stripped of his medal after refusing to cooperate with postmedal drug testing.

CUMULATIVE REVIEW

1. Find the real solutions, if any, of the equation $2x^2 + x - 1 = 0$.
2. Find an equation for the line with slope $-3$ containing the point $(-2, 5)$.
3. Find an equation for a circle of radius 4 and center at the point $(0, -2)$.
4. Discuss the equation $2x - 3y = 12$. Graph it.
5. Discuss the equation $x^2 + y^2 - 2x + 4y - 4 = 0$. Graph it.
6. Use transformations to graph the function $y = (x - 3)^2 + 2$.
7. Sketch a graph of each of the following functions. Label at least three points on each graph.
   (a) $y = x^2$  
   (b) $y = x^3$  
   (c) $y = e^x$  
   (d) $y = \ln x$  
   (e) $y = \sin x$  
   (f) $y = \tan x$
8. Find the inverse function of $f(x) = 3x - 2$.
9. Find the exact value of $(\sin 14^\circ)^2 + (\cos 14^\circ)^2 - 3$.
10. Graph $y = 3 \sin(2x)$.
11. Find the exact value of $\tan \frac{\pi}{4} - 3 \cos \frac{\pi}{6} + \csc \frac{\pi}{6}$.
12. Find an exponential function for the following graph. Express your answer in the form $y = Ab^x$. 

![Graph of function](image)
13. Find a sinusoidal function for the following graph.

\[ y \]

\[ \begin{array}{c}
-6 & 3 & 6 \\
3 & 3 & -3
\end{array} \]

14. (a) Find a linear function that contains the points \((-2, 3)\) and \((1, -6)\). What is the slope? What are the intercepts of the function? Graph the function. Be sure to label the intercepts.

**CHAPTER PROJECTS**

1. **Tides** The given table is a partial tide table for November 2006 for the Sabine Bank Lighthouse, a shoal located offshore from Texas where the Sabine River empties into the Gulf of Mexico.

   1. On November 15, when was the tide high? This is called *high tide*. On November 19, when was the tide low? This is called *low tide*. Most days will have two high tides and two low tides.

   2. Why do you think there is a negative height for the low tide on November 20? What is the height measured against?

3. On your graphing utility, draw a scatter diagram for the data in the table. Let \(t\) (time) be the independent variable, with \(t = 0\) being 12:00 AM on November 14, \(t = 24\) being 12:00 AM on November 15, and so on. Let \(h\) be the height in feet. Remember that there are 60 minutes in an hour. Also, make sure your graphing utility is in radian mode.

4. What shape does the data take? What is the period of the data? What is the amplitude? Is the amplitude constant? Explain.

5. Fit a sine curve to the data. Is there a vertical shift? Is there a phase shift?

6. Using your graphing utility, find the sinusoidal function of best fit. How does this function compare to your equation?

7. Using the equation found in part (5) and the sinusoidal equation of best fit found in part (6), predict the high tides and the low tides on November 21.

8. Looking at the times of day that the low tides occur, what do you think causes the low tides to vary so much each day? Explain. Does this seem to have the same type of effect on the high tides? Explain.

<table>
<thead>
<tr>
<th></th>
<th>Low Tide</th>
<th>Low Tide</th>
<th>High Tide</th>
<th>High Tide</th>
<th>Sun/Moon Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Time</td>
<td>Ht (ft)</td>
<td>Time</td>
<td>Ht (ft)</td>
<td>Ht (ft)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Sunrise/set</td>
</tr>
<tr>
<td>Nov 14</td>
<td>6:26a</td>
<td>2.0</td>
<td>4:38p</td>
<td>1.4</td>
<td>9:29a 2.2</td>
</tr>
<tr>
<td>15</td>
<td>6:22a</td>
<td>1.6</td>
<td>5:34p</td>
<td>1.8</td>
<td>11:18a 2.4</td>
</tr>
<tr>
<td>16</td>
<td>6:28a</td>
<td>1.2</td>
<td>6:25p</td>
<td>2.0</td>
<td>12:37p 2.6</td>
</tr>
<tr>
<td>17</td>
<td>6:40a</td>
<td>0.8</td>
<td>7:12p</td>
<td>2.4</td>
<td>1:38p 2.8</td>
</tr>
<tr>
<td>18</td>
<td>6:56a</td>
<td>0.4</td>
<td>7:57p</td>
<td>2.6</td>
<td>2:27p 3.0</td>
</tr>
<tr>
<td>19</td>
<td>7:17a</td>
<td>0.0</td>
<td>8:38p</td>
<td>2.6</td>
<td>3:10p 3.2</td>
</tr>
<tr>
<td>20</td>
<td>7:43a</td>
<td>-0.2</td>
<td></td>
<td></td>
<td>3:52p 3.4</td>
</tr>
</tbody>
</table>

[Note: a, AM; p, PM.]

**Sources:** National Oceanic and Atmospheric Administration (http://tidesandcurrents.noaa.gov) and U.S. Naval Observatory (http://aa.usno.navy.mil)
CHAPTER 7  Trigonometric Functions

The following projects are available at the Instructor's Resource Center (IRC):

II. Project at Motorola *Digital Transmission over the Air*  Learn how Motorola Corporation transmits digital sequences by modulating the phase of the carrier waves.

III. Identifying Mountain Peaks in Hawaii  The visibility of a mountain is affected by its altitude, distance from the viewer, and the curvature of the earth’s surface. Trigonometry can be used to determine whether a distant object can be seen.

IV. CBL Experiment  Technology is used to model and study the effects of damping on sound waves.