The First Modern Olympics: Athens, 1896

The birth of the modern Olympic Games

By John Gettings—"I hereby proclaim the opening of the first International Olympic Games at Athens." With these words on April 6, 1896, King George I of Greece welcomed the crowd that had gathered in the newly reconstructed Panathenean Stadium to the modern-day Olympic Summer Games.

The event was the idea of Baron Pierre de Coubertin of France who traveled the world to gather support for his dream to have nations come together and overcome national disputes, all in the name of sport.

The program for the Games included track and field, fencing, weightlifting, rifle and pistol shooting, tennis, cycling, swimming, gymnastics, and wrestling. Although 14 nations participated, most of the athletes were Greek.

The Games reached their high point on Day 11 with the first modern-day marathon. The idea to hold an event to commemorate the Ancient Olympic games was suggested by a friend of de Coubertin and was met with great anticipation. The race was run from Marathon to Athens (estimated at 22–26 miles), watched by more than 100,000 people and won by a Greek runner, Spiridon Louis.


—See the Chapter Project—

A Look Back

In Chapter R we reviewed algebra essentials and geometry essentials. In Chapter 1 we studied equations in one variable.

A Look Ahead

Here we connect algebra and geometry using the rectangular coordinate system and use it to graph equations in two variables. The idea of using a system of rectangular coordinates dates back to ancient times, when such a system was used for surveying and city planning. Apollonius of Perga, in 200 BC, used a form of rectangular coordinates in his work on conics, although this use does not stand out as clearly as it does in modern treatments. Sporadic use of rectangular coordinates continued until the 1600s. By that time, algebra has developed sufficiently so that René Descartes (1596–1650) and Pierre de Fermat (1601–1665) could take the crucial step, which was the use of rectangular coordinates to translate geometry problems into algebra problems, and vice versa. This step was important for two reasons. First, it allowed both geometers and algebraists to gain new insights into their subjects, which previously had been regarded as separate, but now were seen to be connected in many important ways. Second, these insights made the development of calculus possible, which greatly enlarged the number of areas in which mathematics could be applied and made possible a much deeper understanding of these areas.
2.1 The Distance and Midpoint Formulas

PREPARING FOR THIS SECTION  
Before getting started, review the following:

• Algebra Essentials (Chapter R, Section R.2, pp. 17–26)
• Geometry Essentials (Chapter R, Section R.3, pp. 30–35)

Now Work the "Are You Prepared?" problems on page 160.

OBJECTIVES  
1 Use the Distance Formula (p. 157)  
2 Use the Midpoint Formula (p. 159)

Rectangular Coordinates

We locate a point on the real number line by assigning it a single real number, called the coordinate of the point. For work in a two-dimensional plane, we locate points by using two numbers.

We begin with two real number lines located in the same plane: one horizontal and the other vertical. We call the horizontal line the \( x \)-axis, the vertical line the \( y \)-axis, and the point of intersection the origin \( O \). See Figure 1. We assign coordinates to every point on these number lines using a convenient scale. We usually use the same scale on each axis. In applications, however, different scales appropriate to the application may be used.

The origin \( O \) has a value of 0 on both the \( x \)-axis and \( y \)-axis. Points on the \( x \)-axis to the right of \( O \) are associated with positive real numbers, and those to the left of \( O \) are associated with negative real numbers. Points on the \( y \)-axis above \( O \) are associated with positive real numbers, and those below \( O \) are associated with negative real numbers. In Figure 1, the \( x \)-axis and \( y \)-axis are labeled as \( x \) and \( y \), respectively, and we have used an arrow at the end of each axis to denote the positive direction.

The coordinate system described here is called a rectangular or Cartesian* coordinate system. The plane formed by the \( x \)-axis and \( y \)-axis is sometimes called the \( xy \)-plane, and the \( x \)-axis and \( y \)-axis are referred to as the coordinate axes.

Any point \( P \) in the \( xy \)-plane can be located by using an ordered pair \((x, y)\) of real numbers. Let \( x \) denote the signed distance of \( P \) from the \( y \)-axis (signed means that, if \( P \) is to the right of the \( y \)-axis, then \( x > 0 \), and if \( P \) is to the left of the \( y \)-axis, then \( x < 0 \)); and let \( y \) denote the signed distance of \( P \) from the \( x \)-axis. The ordered pair \((x, y)\), also called the coordinates of \( P \), then gives us enough information to locate the point \( P \) in the plane.

For example, to locate the point whose coordinates are \((-3, 1)\), go 3 units along the \( x \)-axis to the left of \( O \) and then go straight up 1 unit. We plot this point by placing a dot at this location. See Figure 2, in which the points with coordinates \((-3, 1)\), \((-2, 3)\), \((3, 2)\), and \((-2, -3)\) are plotted.

The origin has coordinates \((0, 0)\). Any point on the \( x \)-axis has coordinates of the form \((x, 0)\), and any point on the \( y \)-axis has coordinates of the form \((0, y)\).

If \((x, y)\) are the coordinates of a point \( P \), then \( x \) is called the \( x \)-coordinate, or abscissa, of \( P \) and \( y \) is the \( y \)-coordinate, or ordinate, of \( P \). We identify the point \( P \) by its coordinates \((x, y)\) by writing \( P = (x, y) \). Usually, we will simply say “the point \((x, y)\)” rather than “the point whose coordinates are \((x, y)\).”

The coordinate axes divide the \( xy \)-plane into four sections called quadrants, as shown in Figure 3. In quadrant I, both the \( x \)-coordinate and the \( y \)-coordinate of all points are positive; in quadrant II, \( x \) is negative and \( y \) is positive; in quadrant III, both \( x \) and \( y \) are negative; and in quadrant IV, \( x \) is positive and \( y \) is negative. Points on the coordinate axes belong to no quadrant.

* Named after René Descartes (1596–1650), a French mathematician, philosopher, and theologian.
COMMENT On a graphing calculator, you can set the scale on each axis. Once this has been done, you obtain the viewing rectangle. See Figure 4 for a typical viewing rectangle. You should now read Section 1, The Viewing Rectangle, in the Appendix.

**EXAMPLE 1**

**Finding the Distance between Two Points**

Find the distance \(d\) between the points \((1, 3)\) and \((5, 6)\).

**Solution**

First we plot the points \((1, 3)\) and \((5, 6)\) and connect them with a straight line. See Figure 5(a). We are looking for the length \(d\). We begin by drawing a horizontal line from \((1, 3)\) to \((5, 3)\) and a vertical line from \((5, 3)\) to \((5, 6)\), forming a right triangle, as shown in Figure 5(b). One leg of the triangle is of length 4 (since \(|5 - 1| = 4\)), and the other is of length 3 (since \(|6 - 3| = 3\)). By the Pythagorean Theorem, the square of the distance \(d\) that we seek is

\[
d^2 = 4^2 + 3^2 = 16 + 9 = 25
\]

\[
d = \sqrt{25} = 5
\]

The distance formula provides a straightforward method for computing the distance between two points.

**THEOREM**

**Distance Formula**

The distance between two points \(P_1 = (x_1, y_1)\) and \(P_2 = (x_2, y_2)\), denoted by \(d(P_1, P_2)\), is

\[
d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]  \(\text{(1)}\)

**Proof of the Distance Formula** Let \((x_1, y_1)\) denote the coordinates of point \(P_1\) and let \((x_2, y_2)\) denote the coordinates of point \(P_2\). Assume that the line joining \(P_1\) and \(P_2\) is neither horizontal nor vertical. Refer to Figure 6(a). The coordinates of \(P_3\) are \((x_2, y_1)\). The horizontal distance from \(P_1\) to \(P_3\) is the absolute value of
CHAPTER 2 Graphs

the difference of the x-coordinates, \(|x_2 - x_1|\). The vertical distance from \(P_3\) to \(P_2\) is the absolute value of the difference of the y-coordinates, \(|y_2 - y_1|\). See Figure 6(b). The distance \(d(P_1, P_2)\) that we seek is the length of the hypotenuse of the right triangle, so, by the Pythagorean Theorem, it follows that

\[
[d(P_1, P_2)]^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2
= (x_2 - x_1)^2 + (y_2 - y_1)^2
\]

\[d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\]

![Figure 6](image)

Now, if the line joining \(P_1\) and \(P_2\) is horizontal, then the y-coordinate of \(P_1\) equals the y-coordinate of \(P_2\); that is, \(y_1 = y_2\). Refer to Figure 7(a). In this case, the distance formula (1) still works, because, for \(y_1 = y_2\), it reduces to

\[d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + 0^2} = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|\]

![Figure 7](image)

A similar argument holds if the line joining \(P_1\) and \(P_2\) is vertical. See Figure 7(b). The distance formula is valid in all cases.

**EXAMPLE 2** Using the Distance Formula

Find the distance \(d\) between the points \((-4, 5)\) and \((3, 2)\).

**Solution**

Using the distance formula, equation (1), the solution is obtained as follows:

\[
d = \sqrt{[3 - (-4)]^2 + (2 - 5)^2} = \sqrt{7^2 + (-3)^2}
= \sqrt{49 + 9} = \sqrt{58} \approx 7.62
\]

**Now Work** PROBLEMS 15 AND 19

The distance between two points \(P_1 = (x_1, y_1)\) and \(P_2 = (x_2, y_2)\) is never a negative number. Furthermore, the distance between two points is 0 only when the points are identical, that is, when \(x_1 = x_2\) and \(y_1 = y_2\). Also, because \((x_2 - x_1)^2 = (x_1 - x_2)^2\) and \((y_2 - y_1)^2 = (y_1 - y_2)^2\), it makes no difference whether the distance is computed from \(P_1\) to \(P_2\) or from \(P_2\) to \(P_1\); that is, \(d(P_1, P_2) = d(P_2, P_1)\).

The introduction to this chapter mentioned that rectangular coordinates enable us to translate geometry problems into algebra problems, and vice versa. The next example shows how algebra (the distance formula) can be used to solve geometry problems.
EXAMPLE 3

Using Algebra to Solve Geometry Problems

Consider the three points $A = (-2, 1)$, $B = (2, 3)$, and $C = (3, 1)$.

(a) Plot each point and form the triangle $ABC$.
(b) Find the length of each side of the triangle.
(c) Verify that the triangle is a right triangle.
(d) Find the area of the triangle.

Solution

(a) Points $A$, $B$, and $C$ and triangle $ABC$ are plotted in Figure 8.

(b) We use the distance formula, equation (1).

\[
\begin{align*}
d(A, B) &= \sqrt{[2 - (-2)]^2 + (3 - 1)^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5} \\
d(B, C) &= \sqrt{(3 - 2)^2 + (1 - 3)^2} = \sqrt{1 + 4} = \sqrt{5} \\
d(A, C) &= \sqrt{(3 - (-2))^2 + (1 - 1)^2} = \sqrt{25 + 0} = 5
\end{align*}
\]

(c) To show that the triangle is a right triangle, we need to show that the sum of the squares of the lengths of two of the sides equals the square of the length of the third side. (Why is this sufficient?) Looking at Figure 8, it seems reasonable to conjecture that the right angle is at vertex $B$. We shall check to see whether

\[
[d(A, B)]^2 + [d(B, C)]^2 = [d(A, C)]^2
\]

We find that

\[
[d(A, B)]^2 + [d(B, C)]^2 = (2\sqrt{5})^2 + (\sqrt{5})^2 = 20 + 5 = 25 = [d(A, C)]^2
\]

so it follows from the converse of the Pythagorean Theorem that triangle $ABC$ is a right triangle.

(d) Because the right angle is at vertex $B$, the sides $AB$ and $BC$ form the base and height of the triangle. Its area is

\[
\text{Area} = \frac{1}{2} \times \text{(Base)} \times \text{(Height)} = \frac{1}{2} \times (2\sqrt{5}) \times (\sqrt{5}) = 5 \text{ square units}
\]

Now Work

PROBLEM 29

2 Use the Midpoint Formula

We now derive a formula for the coordinates of the midpoint of a line segment. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be the endpoints of a line segment, and let $M = (x, y)$ be the point on the line segment that is the same distance from $P_1$ as it is from $P_2$. See Figure 9. The triangles $P_1AM$ and $MBP_2$ are congruent. Do you see why? $d(P_1, M) = d(M, P_2)$ is given; $\angle AP_1M = \angle BMP_2$, and $\angle P_1MA = \angle MP_2B$. So, we have angle-side-angle. Because triangles $P_1AM$ and $MBP_2$ are congruent, corresponding sides are equal in length. That is,

\[
\begin{align*}
x - x_1 &= x_2 - x \\
2x &= x_1 + x_2 \\
x &= \frac{x_1 + x_2}{2}
\end{align*}
\]

\[
\begin{align*}
y - y_1 &= y_2 - y \\
2y &= y_1 + y_2 \\
y &= \frac{y_1 + y_2}{2}
\end{align*}
\]

* A postulate from geometry states that the transversal $P_1P_2$ forms congruent corresponding angles with the parallel line segments $P_1A$ and $MB$. 
**Midpoint Formula**

The midpoint \( M = (x, y) \) of the line segment from \( P_1 = (x_1, y_1) \) to \( P_2 = (x_2, y_2) \) is

\[
M = (x, y) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

**EXAMPLE 4**

**Finding the Midpoint of a Line Segment**

Find the midpoint of a line segment from \( P_1 = (-5, 5) \) to \( P_2 = (3, 1) \). Plot the points \( P_1 \) and \( P_2 \) and their midpoint. Check your answer.

**Solution**

We apply the midpoint formula (2) using \( x_1 = -5, y_1 = 5, x_2 = 3, \) and \( y_2 = 1 \). Then the coordinates \((x, y)\) of the midpoint \( M \) are

\[
x = \frac{x_1 + x_2}{2} = \frac{-5 + 3}{2} = -1 \quad \text{and} \quad y = \frac{y_1 + y_2}{2} = \frac{5 + 1}{2} = 3
\]

That is, \( M = (-1, 3) \). See Figure 10.

**Check:** Because \( M \) is the midpoint, we check the answer by verifying that \( d(P_1, M) = d(M, P_2) \):

\[
d(P_1, M) = \sqrt{(-1 - (-5))^2 + (3 - 5)^2} = \sqrt{16 + 4} = \sqrt{20}
\]

\[
d(M, P_2) = \sqrt{(3 - (-1))^2 + (1 - 3)^2} = \sqrt{16 + 4} = \sqrt{20}
\]

**2.1 Assess Your Understanding**

**‘Are You Prepared?’** Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. On the real number line the origin is assigned the number ______. (p. 17–26)
2. If \(-3\) and \(5\) are the coordinates of two points on the real number line, the distance between these points is ______. (p. 17–26)
3. If \(3\) and \(4\) are the legs of a right triangle, the hypotenuse is ______. (pp. 30–35)
4. Use the converse of the Pythagorean Theorem to show that a triangle whose sides are of lengths \(11, 60,\) and \(61\) is a right triangle. (p. 30–35)

**Concepts and Vocabulary**

5. If \((x, y)\) are the coordinates of a point \(P\) in the \(xy\)-plane, then \(x\) is called the _____ of \(P\) and \(y\) is the _____ of \(P\).
6. The coordinate axes divide the \(xy\)-plane into four sections called _____.
7. If three distinct points \(P, Q,\) and \(R\) all lie on a line and if \(d(P, Q) = d(Q, R)\), then \(Q\) is called the _____ of the line segment from \(P\) to \(R\).
8. **True or False** The distance between two points is sometimes a negative number.
9. **True or False** The point \((-1, 4)\) lies in quadrant IV of the Cartesian plane.
10. **True or False** The midpoint of a line segment is found by averaging the \(x\)-coordinates and averaging the \(y\)-coordinates of the endpoints.
Skill Building

In Problems 11 and 12, plot each point in the xy-plane. Tell in which quadrant or on what coordinate axis each point lies.

11. (a) \( A = (-3, 2) \) (d) \( D = (6, 5) \)
    (b) \( B = (6, 0) \) (e) \( E = (0, -3) \)
    (c) \( C = (-2, -2) \) (f) \( F = (6, -3) \)

12. (a) \( A = (1, 4) \) (d) \( D = (4, 1) \)
    (b) \( B = (-3, -4) \) (e) \( E = (0, 1) \)
    (c) \( C = (-3, 4) \) (f) \( F = (-3, 0) \)

13. Plot the points \( (2, 0), \ (2, -3), \ (2, 4), \ (2, 1), \) and \( (2, -1) \). Describe the set of all points of the form \( (2, y) \), where \( y \) is a real number.

14. Plot the points \( (0, 3), \ (1, 3), \ (-2, 3), \ (5, 3), \) and \( (-4, 3) \). Describe the set of all points of the form \( (x, 3) \), where \( x \) is a real number.

In Problems 15–28, find the distance \( d(P_1, P_2) \) between the points \( P_1 \) and \( P_2 \).

15. \( P_1 = (0, 0); \ P_2 = (2, 1) \)
16. \( P_1 = (0, 0); \ P_2 = (-2, 1) \)
17. \( P_1 = (1, 1); \ P_2 = (-2, 2) \)
18. \( P_1 = (-1, 1); \ P_2 = (2, 2) \)
19. \( P_1 = (3, -4); \ P_2 = (5, 4) \)
20. \( P_1 = (-1, 0); \ P_2 = (2, 4) \)
21. \( P_1 = (-3, 2); \ P_2 = (6, 0) \)
22. \( P_1 = (2, -3); \ P_2 = (4, 2) \)
23. \( P_1 = (4, -3); \ P_2 = (6, 4) \)
24. \( P_1 = (-4, -3); \ P_2 = (6, 2) \)
25. \( P_1 = (-0.2, 0.3); \ P_2 = (2.3, 1.1) \)
26. \( P_1 = (1.2, 2.3); \ P_2 = (-0.3, 1.1) \)
27. \( P_1 = (a, b); \ P_2 = (0, 0) \)
28. \( P_1 = (a, a); \ P_2 = (0, 0) \)

In Problems 29–34, plot each point and form the triangle \( ABC \). Verify that the triangle is a right triangle. Find its area.

29. \( A = (-2, 5); \ B = (1, 3); \ C = (-1, 0) \)
30. \( A = (-2, 5); \ B = (12, 3); \ C = (10, -11) \)
31. \( A = (-5, 3); \ B = (6, 0); \ C = (5, 5) \)
32. \( A = (-6, 3); \ B = (3, -5); \ C = (-1, 5) \)
33. \( A = (4, -3); \ B = (0, -3); \ C = (4, 2) \)
34. \( A = (4, 3); \ B = (4, 1); \ C = (2, 1) \)

In Problems 35–44, find the midpoint of the line segment joining the points \( P_1 \) and \( P_2 \).

35. \( P_1 = (3, -4); \ P_2 = (5, 4) \)
36. \( P_1 = (-2, 0); \ P_2 = (2, 4) \)
37. \( P_1 = (-3, 2); \ P_2 = (6, 0) \)
38. \( P_1 = (2, -3); \ P_2 = (4, 2) \)
39. \( P_1 = (4, -3); \ P_2 = (6, 1) \)
40. \( P_1 = (-4, -3); \ P_2 = (2, 2) \)
41. \( P_1 = (-0.2, 0.3); \ P_2 = (2.3, 1.1) \)
42. \( P_1 = (1.2, 2.3); \ P_2 = (-0.3, 1.1) \)
43. \( P_1 = (a, b); \ P_2 = (0, 0) \)
44. \( P_1 = (a, a); \ P_2 = (0, 0) \)

Applications and Extensions

45. Find all points having an \( x \)-coordinate of 2 whose distance from the point \( (-2, -1) \) is 5.
46. Find all points having a \( y \)-coordinate of -3 whose distance from the point \( (1, 2) \) is 13.
47. Find all points on the \( x \)-axis that are 5 units from the point \((4, -3)\).
48. Find all points on the \( y \)-axis that are 5 units from the point \((4, 4)\).
49. The midpoint of the line segment from \( P_1 \) to \( P_2 \) is \((-1, 4)\). If \( P_1 = (-3, 6) \), what is \( P_2 \)?
50. The midpoint of the line segment from \( P_1 \) to \( P_2 \) is \((5, -4)\). If \( P_2 = (7, -2) \), what is \( P_1 \)?
51. Geometry The medians of a triangle are the line segments from each vertex to the midpoint of the opposite side (see the figure). Find the lengths of the medians of the triangle with vertices at \( A = (0, 0), B = (6, 0), \) and \( C = (4, 4) \).
52. **Geometry** An equilateral triangle is one in which all three sides are of equal length. If two vertices of an equilateral triangle are \((0, 4)\) and \((0, 0)\), find the third vertex. How many of these triangles are possible?

\[ \begin{align*}
&\text{s} \\
&\text{s} \\
&\text{s}
\end{align*} \]

53. **Geometry** Find the midpoint of each diagonal of a square with side of length \(s\). Draw the conclusion that the diagonals of a square intersect at their midpoints.

[Hint: Use \((0, 0), (0, s), (s, 0),\) and \((s, s)\) as the vertices of the square.]

54. **Geometry** Verify that the points \((0, 0), (a, 0),\) and \((a, \sqrt{3}a/2)\) are the vertices of an equilateral triangle. Then show that the midpoints of the three sides are the vertices of a second equilateral triangle (refer to Problem 52).

In Problems 55–58, find the length of each side of the triangle determined by the three points and state whether the triangle is an isosceles triangle, a right triangle, neither of these, or both. (An isosceles triangle is one in which at least two of the sides are of equal length.)

55. \(P_1 = (2, 1);\ P_2 = (-4, 1);\ P_3 = (-4, -3)\)
56. \(P_1 = (-1, -4);\ P_2 = (6, 2);\ P_3 = (4, -5)\)
57. \(P_1 = (-2, -1);\ P_2 = (0, 7);\ P_3 = (3, 2)\)
58. \(P_1 = (7, 2);\ P_2 = (-4, 0);\ P_3 = (4, 6)\)
59. **Baseball** A major league baseball “diamond” is actually a square, 90 feet on a side (see the figure). What is the distance directly from home plate to second base (the diagonal of the square)?

**60. Little League Baseball** The layout of a Little League playing field is a square, 60 feet on a side. How far is it directly from home plate to second base (the diagonal of the square)?


**61. Baseball** Refer to Problem 59. Overlay a rectangular coordinate system on a major league baseball diamond so that the origin is at home plate, the positive x-axis lies in the direction from home plate to first base, and the positive y-axis lies in the direction from home plate to third base.

(a) What are the coordinates of first base, second base, and third base? Use feet as the unit of measurement.

(b) If the right fielder is located at \((310, 15)\), how far is it from the right fielder to second base?

(c) If the center fielder is located at \((300, 300)\), how far is it from the center fielder to third base?

**62. Little League Baseball** Refer to Problem 60. Overlay a rectangular coordinate system on a Little League baseball diamond so that the origin is at home plate, the positive x-axis lies in the direction from home plate to first base, and the positive y-axis lies in the direction from home plate to third base.

(a) What are the coordinates of first base, second base, and third base? Use feet as the unit of measurement.

(b) If the right fielder is located at \((180, 20)\), how far is it from the right fielder to second base?

(c) If the center fielder is located at \((220, 220)\), how far is it from the center fielder to third base?

**63. Distance between Moving Objects** A Dodge Neon and a Mack truck leave an intersection at the same time. The Neon heads east at an average speed of 30 miles per hour, while the truck heads south at an average speed of 40 miles per hour. Find an expression for their distance apart \(d\) (in miles) at the end of \(t\) hours.

**64. Distance of a Moving Object from a Fixed Point** A hot-air balloon, headed due east at an average speed of 15 miles per hour and at a constant altitude of 100 feet, passes over an intersection (see the figure). Find an expression for the distance \(d\) (measured in feet) from the balloon to the intersection \(t\) seconds later.

**65. Drafting Error** When a draftsman draws three lines that are to intersect at one point, the lines may not intersect as intended and subsequently will form an error triangle. If this error triangle is long and thin, one estimate for the location of the desired point is the midpoint of the shortest side. The figure shows such an error triangle.

Source: www.uwgb.edu/dutchs/STRUCTGE/s00.htm
2.2 Graphs of Equations in Two Variables; Intercepts; Symmetry

PREPARING FOR THIS SECTION
Before getting started, review the following:
- Solving Equations (Section 1.1, pp. 86–93)

Now Work the ‘Are You Prepared?’ problems on page 171.

OBJECTIVES
1. Graph Equations by Plotting Points (p. 163)
2. Find Intercepts from a Graph (p. 165)
3. Find Intercepts from an Equation (p. 166)
4. Test an Equation for Symmetry with Respect to the x-Axis, the y-Axis, and the Origin (p. 167)
5. Know How to Graph Key Equations (p. 169)

Graph Equations by Plotting Points
An equation in two variables, say \(x\) and \(y\), is a statement in which two expressions involving \(x\) and \(y\) are equal. The expressions are called the sides of the equation. Since an equation is a statement, it may be true or false, depending on the value of the variables. Any values of \(x\) and \(y\) that result in a true statement are said to satisfy the equation.

For example, the following are all equations in two variables \(x\) and \(y\):

\[
\begin{align*}
\text{(1.4, 1.3)} & \quad (2.7, 1.7) & \quad (2.6, 1.5) \\
1.7 & \quad 1.5 & \quad 1.3 \\
\end{align*}
\]

(a) Find an estimate for the desired intersection point.
(b) Find the length of the median for the midpoint found in part (a). See Problem 51.

66. Net Sales The figure illustrates how net sales of Wal-Mart Stores, Inc., have grown from 2002 through 2006. Use the midpoint formula to estimate the net sales of Wal-Mart Stores, Inc., in 2004. How does your result compare to the reported value of $256 billion?


‘Are You Prepared?’ Answers
1. 0 2. 8 3. 5 4. \(11^2 + 60^2 = 121 + 3600 = 3721 = 61^2\)

SULLMC02_013157759X.QXD 11/28/06 1:19 PM Page 163
gasoline in California based on 2005 dollars (adjusted for inflation) for the years 1978–2005. In Figure 11, we graph the data from the table, using the year along the x-axis and the price along the y-axis. From the graph, we can see the price was highest in 1980, 1981, and 2005 at about $2.50 per gallon.

### Example 1

**Determining Whether a Point Is on the Graph of an Equation**

Determine if the following points are on the graph of the equation $2x - y = 6$.

(a) $(2, 3)$  
(b) $(2, -2)$

**Solution**

(a) For the point $(2, 3)$, we check to see if $x = 2, y = 3$ satisfies the equation $2x - y = 6$.

$$2x - y = 2(2) - 3 = 4 - 3 = 1 \neq 6$$

The equation is not satisfied, so the point $(2, 3)$ is not on the graph of $2x - y = 6$.

(b) For the point $(2, -2)$, we have

$$2x - y = 2(2) - (-2) = 4 + 2 = 6$$

The equation is satisfied, so the point $(2, -2)$ is on the graph of $2x - y = 6$.

### Example 2

**Graphing an Equation by Plotting Points**

Graph the equation: $y = 2x + 5$

**Solution**

We want to find all points $(x, y)$ that satisfy the equation. To locate some of these points (and get an idea of the pattern of the graph), we assign some numbers to $x$ and find corresponding values for $y$.

<table>
<thead>
<tr>
<th>If</th>
<th>Then</th>
<th>Point on Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>$y = 2(0) + 5 = 5$</td>
<td>(0, 5)</td>
</tr>
<tr>
<td>$x = 1$</td>
<td>$y = 2(1) + 5 = 7$</td>
<td>(1, 7)</td>
</tr>
<tr>
<td>$x = -5$</td>
<td>$y = 2(-5) + 5 = -5$</td>
<td>(-5, -5)</td>
</tr>
<tr>
<td>$x = 10$</td>
<td>$y = 2(10) + 5 = 25$</td>
<td>(10, 25)</td>
</tr>
</tbody>
</table>

By plotting these points and then connecting them, we obtain the graph of the equation (a line), as shown in Figure 12.
Graphing an Equation by Plotting Points

Graph the equation: \( y = x^2 \)

**Solution**

Table 2 provides several points on the graph. In Figure 13 we plot these points and connect them with a smooth curve to obtain the graph (a parabola).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^2 )</th>
<th>( (x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>16</td>
<td>(-4, 16)</td>
</tr>
<tr>
<td>-3</td>
<td>9</td>
<td>(-3, 9)</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>(-2, 4)</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>(3, 9)</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>(4, 16)</td>
</tr>
</tbody>
</table>

The graphs of the equations shown in Figures 12 and 13 do not show all points. For example, in Figure 12, the point (20, 45) is a part of the graph of \( y = 2x + 5 \), but it is not shown. Since the graph of \( y = 2x + 5 \) could be extended out as far as we please, we use arrows to indicate that the pattern shown continues. It is important when illustrating a graph to present enough of the graph so that any viewer of the illustration will “see” the rest of it as an obvious continuation of what is actually there. This is referred to as a **complete graph**.

One way to obtain a complete graph of an equation is to plot a sufficient number of points on the graph until a pattern becomes evident. Then these points are connected with a smooth curve following the suggested pattern. But how many points are sufficient? Sometimes knowledge about the equation tells us. For example, we will learn in the next section that, if an equation is of the form \( y = mx + b \), then its graph is a line. In this case, only two points are needed to obtain the graph.

One purpose of this book is to investigate the properties of equations in order to decide whether a graph is complete. Sometimes we shall graph equations by plotting points. Shortly, we shall investigate various techniques that will enable us to graph an equation without plotting so many points.

**COMMENT** Another way to obtain the graph of an equation is to use a graphing utility. Read Section 2, Using a Graphing Utility to Graph Equations, in the Appendix.

Two techniques that sometimes reduce the number of points required to graph an equation involve finding **intercepts** and checking for **symmetry**.

2 Find Intercepts from a Graph

The points, if any, at which a graph crosses or touches the coordinate axes are called the **intercepts**. See Figure 14. The \( x \)-coordinate of a point at which the graph crosses or touches the \( x \)-axis is an **\( x \)-intercept**, and the \( y \)-coordinate of a point at which the graph crosses or touches the \( y \)-axis is a **\( y \)-intercept**. For a graph to be complete, all its intercepts must be displayed.

**EXAMPLE 4** Finding Intercepts from a Graph

Find the intercepts of the graph in Figure 15 shown on p. 166. What are its \( x \)-intercepts? What are its \( y \)-intercepts?
The intercepts of the graph are the points 

\((-3, 0), \ (0, 3), \ (3/2, 0), \ (0, -4/3), \ (0, -3.5), \ (4.5, 0)\)

The \(x\)-intercepts are \(-3, 3/2, \) and \(4.5\); the \(y\)-intercepts are \(-3.5, \ -4/3, \) and \(3\).

In Example 4, you should notice the following usage: If we do not specify the type of intercept (\(x\)- versus \(y\)-), then we report the intercept as an ordered pair. However, if we specify the type of intercept, then we only report the coordinate of the intercept. For \(x\)-intercepts, we report the \(x\)-coordinate of the intercept; for \(y\)-intercepts, we report the \(y\)-coordinate of the intercept.

**Now Work Problem 39(a)**

### 3 Find Intercepts from an Equation

The intercepts of a graph can be found from its equation by using the fact that points on the \(x\)-axis have \(y\)-coordinates equal to 0 and points on the \(y\)-axis have \(x\)-coordinates equal to 0.

**Procedure for Finding Intercepts**

1. To find the \(x\)-intercept(s), if any, of the graph of an equation, let \(y = 0\) in the equation and solve for \(x\), where \(x\) is a real number.
2. To find the \(y\)-intercept(s), if any, of the graph of an equation, let \(x = 0\) in the equation and solve for \(y\), where \(y\) is a real number.

**Finding Intercepts from an Equation**

Find the \(x\)-intercept(s) and the \(y\)-intercept(s) of the graph of \(y = x^2 - 4\). Then graph \(y = x^2 - 4\) by plotting points.

**Solution**

To find the \(x\)-intercept(s), we let \(y = 0\) and obtain the equation

\[ x^2 - 4 = 0 \]

\[ x + 2 = 0 \quad \text{or} \quad x - 2 = 0 \]

\[ x = -2 \quad \text{or} \quad x = 2 \]

The equation has two solutions, \(-2\) and \(2\). The \(x\)-intercepts are \(-2\) and \(2\).

To find the \(y\)-intercept(s), we let \(x = 0\) in the equation.

\[ y = x^2 - 4 \]

\[ = 0^2 - 4 = -4 \]

The \(y\)-intercept is \(-4\).

Since \(x^2 \geq 0\) for all \(x\), we deduce from the equation \(y = x^2 - 4\) that \(y \geq -4\) for all \(x\). This information, the intercepts, and the points from Table 3 enable us to graph \(y = x^2 - 4\). See Figure 16.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y = x^2 - 4)</th>
<th>((x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3)</td>
<td>5</td>
<td>((-3, 5))</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-3)</td>
<td>((-1, -3))</td>
</tr>
<tr>
<td>1</td>
<td>(-3)</td>
<td>((1, -3))</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>((3, 5))</td>
</tr>
</tbody>
</table>

**Now Work Problem 21**
SECTION 2.2 Graphs of Equations in Two Variables; Intercepts; Symmetry

4 Test an Equation for Symmetry with Respect to the x-Axis, the y-Axis, and the Origin

We just saw the role that intercepts play in obtaining key points on the graph of an equation. Another helpful tool for graphing equations involves symmetry, particularly symmetry with respect to the x-axis, the y-axis, and the origin.

**DEFINITION**

A graph is said to be **symmetric with respect to the x-axis** if, for every point \( (x, y) \) on the graph, the point \( (x, -y) \) is also on the graph.

Figure 17 illustrates the definition. When a graph is symmetric with respect to the x-axis, notice that the part of the graph above the x-axis is a reflection or mirror image of the part below it, and vice versa.

**EXAMPLE 6**

Points Symmetric with Respect to the x-Axis

If a graph is symmetric with respect to the x-axis and the point \( (3, 2) \) is on the graph, then the point \( (3, -2) \) is also on the graph.

**DEFINITION**

A graph is said to be **symmetric with respect to the y-axis** if, for every point \( (x, y) \) on the graph, the point \( (-x, y) \) is also on the graph.

Figure 18 illustrates the definition. When a graph is symmetric with respect to the y-axis, notice that the part of the graph to the right of the y-axis is a reflection of the part to the left of it, and vice versa.

**EXAMPLE 7**

Points Symmetric with Respect to the y-Axis

If a graph is symmetric with respect to the y-axis and the point \( (5, 8) \) is on the graph, then the point \( (-5, 8) \) is also on the graph.

**DEFINITION**

A graph is said to be **symmetric with respect to the origin** if, for every point \( (x, y) \) on the graph, the point \( (-x, -y) \) is also on the graph.

Figure 19 illustrates the definition. Notice that symmetry with respect to the origin may be viewed in three ways:

1. As a reflection about the y-axis, followed by a reflection about the x-axis
2. As a projection along a line through the origin so that the distances from the origin are equal
3. As half of a complete revolution about the origin
EXAMPLE 8

Points Symmetric with Respect to the Origin

If a graph is symmetric with respect to the origin and the point (4, 2) is on the graph, then the point (−4, −2) is also on the graph.

Now Work PROBLEMS 29 AND 39(b)

When the graph of an equation is symmetric with respect to a coordinate axis or the origin, the number of points that you need to plot in order to see the pattern is reduced. For example, if the graph of an equation is symmetric with respect to the y-axis, then, once points to the right of the y-axis are plotted, an equal number of points on the graph can be obtained by reflecting them about the y-axis. Because of this, before we graph an equation, we first want to determine whether it has any symmetry. The following tests are used for this purpose.

Tests for Symmetry

To test the graph of an equation for symmetry with respect to the

- x-Axis: Replace y by −y in the equation and simplify. If an equivalent equation results, the graph of the equation is symmetric with respect to the x-axis.
- y-Axis: Replace x by −x in the equation and simplify. If an equivalent equation results, the graph of the equation is symmetric with respect to the y-axis.
- Origin: Replace x by −x and y by −y in the equation and simplify. If an equivalent equation results, the graph of the equation is symmetric with respect to the origin.

EXAMPLE 9

Testing an Equation for Symmetry

Test \( y = \frac{4x^2}{x^2 + 1} \) for symmetry.

Solution

- x-Axis: To test for symmetry with respect to the x-axis, replace y by −y. Since
  \[ -y = \frac{4(-x)^2}{(-x)^2 + 1} = \frac{4x^2}{x^2 + 1} \]
  is not equivalent to \( y = \frac{4x^2}{x^2 + 1} \), the graph of the equation is not symmetric with respect to the x-axis.

- y-Axis: To test for symmetry with respect to the y-axis, replace x by −x. Since
  \[ y = \frac{4(-x)^2}{(-x)^2 + 1} = \frac{4x^2}{x^2 + 1} \]
  is equivalent to \( y = \frac{4x^2}{x^2 + 1} \), the graph of the equation is symmetric with respect to the y-axis.

- Origin: To test for symmetry with respect to the origin, replace x by −x and y by −y.
  \[ -y = \frac{4(-x)^2}{(-x)^2 + 1} \]
  Replace x by −x and y by −y.
  \[ -y = \frac{4x^2}{x^2 + 1} \]
  Simplify.
  \[ y = -\frac{4x^2}{x^2 + 1} \]
  Multiply both sides by −1.

Since the result is not equivalent to the original equation, the graph of the equation \( y = \frac{4x^2}{x^2 + 1} \) is not symmetric with respect to the origin.
SECTION 2.2  Graphs of Equations in Two Variables; Intercepts; Symmetry

Seeing the Concept

Figure 20 shows the graph of \( y = \frac{4x^3}{x^2 + 1} \) using a graphing utility. Do you see the symmetry with respect to the \( y \)-axis?

\[
\begin{align*}
\text{Figure 20} & \quad \begin{array}{c}
\begin{array}{c}
\text{5} \\
\text{Now Work PROBLEM 59}
\end{array}
\end{array}
\end{align*}
\]

Now Work  PROBLEM 59

5 Know How to Graph Key Equations

The next three examples use intercepts, symmetry, and point plotting to obtain the graphs of key equations. It is important to know the graphs of these key equations because we use them later. The first of these is \( y = x^3 \).

EXAMPLE 10  Graphing the Equation \( y = x^3 \) by Finding Intercepts, Checking for Symmetry, and Plotting Points

Graph the equation \( y = x^3 \) by plotting points. Find any intercepts and check for symmetry first.

Solution
First, we seek the intercepts. When \( x = 0 \), then \( y = 0 \); and when \( y = 0 \), then \( x = 0 \). The origin \((0, 0)\) is the only intercept. Now we test for symmetry.

- \( x \)-Axis: Replace \( y \) by \(-y\). Since \(-y = x^3\) is not equivalent to \( y = x^3 \), the graph is not symmetric with respect to the \( x \)-axis.
- \( y \)-Axis: Replace \( x \) by \(-x\). Since \( y = (-x)^3 = -x^3 \) is not equivalent to \( y = x^3 \), the graph is not symmetric with respect to the \( y \)-axis.
- Origin: Replace \( x \) by \(-x\) and \( y \) by \(-y\). Since \(-y = (-x)^3 = -x^3 \) is equivalent to \( y = x^3 \) (multiply both sides by \(-1\)), the graph is symmetric with respect to the origin.

To graph \( y = x^3 \), we use the equation to obtain several points on the graph. Because of the symmetry, we only need to locate points on the graph for which \( x \geq 0 \). See Table 4. Since \((1, 1)\) is on the graph, and the graph is symmetric with respect to the origin, the point \((-1, -1)\) is also on the graph. Plot the points from Table 4. Figure 21 shows the graph.

\[
\begin{array}{|c|c|c|}
\hline
x & y = x^3 & (x, y) \\
\hline
0 & 0 & (0, 0) \\
1 & 1 & (1, 1) \\
2 & 8 & (2, 8) \\
3 & 27 & (3, 27) \\
\hline
\end{array}
\]

\[
\begin{align*}
\text{Table 4} & \\
\text{Figure 21} & \\
\end{align*}
\]
Graphing the Equation \( x = y^2 \)

Graph the equation \( x = y^2 \). Find any intercepts and check for symmetry first.

**Solution**

The lone intercept is \((0, 0)\). The graph is symmetric with respect to the \(x\)-axis. (Do you see why? Replace \(y\) by \(-y\).) Figure 22 shows the graph.

If we restrict \(y\) so that \(y \geq 0\), the equation \(x = y^2\), \(y \geq 0\), may be written equivalently as \(y = \sqrt{x}\). The portion of the graph of \(x = y^2\) in quadrant I is therefore the graph of \(y = \sqrt{x}\). See Figure 23.

**COMMENT**

To see the graph of the equation \(x = y^2\) on a graphing calculator, you will need to graph two equations: \(Y_1 = \sqrt{x}\) and \(Y_2 = -\sqrt{x}\). We discuss why in Chapter 3. See Figure 24.

---

Graphing the Equation \( y = \frac{1}{x} \)

Graph the equation \(y = \frac{1}{x}\). Find any intercepts and check for symmetry first.

**Solution**

We check for intercepts first. If we let \(x = 0\), we obtain 0 in the denominator, which makes \(y\) undefined. We conclude that there is no \(y\)-intercept. If we let \(y = 0\), we get the equation \(\frac{1}{x} = 0\), which has no solution. We conclude that there is no \(x\)-intercept. The graph of \(y = \frac{1}{x}\) does not cross or touch the coordinate axes.

Next we check for symmetry:

- **\(x\)-Axis:** Replacing \(y\) by \(-y\) yields \(-y = \frac{1}{x}\), which is not equivalent to \(y = \frac{1}{x}\).

- **\(y\)-Axis:** Replacing \(x\) by \(-x\) yields \(y = \frac{1}{-x} = -\frac{1}{x}\), which is not equivalent to \(y = \frac{1}{x}\).

- **Origin:** Replacing \(x\) by \(-x\) and \(y\) by \(-y\) yields \(-y = -\frac{1}{x}\), which is equivalent to \(y = \frac{1}{x}\). The graph is symmetric with respect to the origin.

Now, we set up Table 5, listing several points on the graph. Because of the symmetry with respect to the origin, we use only positive values of \(x\). From Table 5 we
2.2 Assess Your Understanding

‘Are You Prepared?’ Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Solve the equation \(2x + 3 - 1 = -7\). (pp. 86–93)

2. Solve the equation \(x^2 - 9 = 0\). (p. 86–93)

**Concepts and Vocabulary**

3. The points, if any, at which a graph crosses or touches the coordinate axes are called _________.

4. The \(x\)-intercepts of the graph of an equation are those \(x\)-values for which _________.

5. If for every point \((x, y)\) on the graph of an equation the point \((-x, y)\) is also on the graph, then the graph is symmetric with respect to the _________.

6. If the graph of an equation is symmetric with respect to the \(y\)-axis and \(-4\) is an \(x\)-intercept of this graph, then _________ is also an \(x\)-intercept.

7. If the graph of an equation is symmetric with respect to the origin and \((3, \ -4)\) is a point on the graph, then _________ is also a point on the graph.

8. True or False To find the \(y\)-intercepts of the graph of an equation, let \(x = 0\) and solve for \(y\).

9. True or False The \(y\)-coordinate of a point at which the graph crosses or touches the \(x\)-axis is an \(x\)-intercept.

10. True or False If a graph is symmetric with respect to the \(x\)-axis, then it cannot be symmetric with respect to the \(y\)-axis.

**Skill Building**

**In Problems 11–16, determine which of the given points are on the graph of the equation.**

<table>
<thead>
<tr>
<th></th>
<th>Equation: (y = x^4 - \sqrt{x})</th>
<th>Points: ((0, 0); \ (1, 1); \ (-1, 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>(y = x^3 - 2\sqrt{x})</td>
<td>Points: ((0, 0); \ (1, 1); \ (1, -1))</td>
</tr>
<tr>
<td>12</td>
<td>(y = x^3 + y^2 = 4)</td>
<td>Points: ((0, 2); \ (-2, 2); \ (\sqrt{2}, \sqrt{2}))</td>
</tr>
</tbody>
</table>

**In Problems 17–28, find the intercepts and graph each equation by plotting points. Be sure to label the intercepts.**

<table>
<thead>
<tr>
<th></th>
<th>(y = x + 2)</th>
<th>(y = x - 6)</th>
<th>(y = 2x + 8)</th>
<th>(y = 3x - 9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>(y = x^2 - 1)</td>
<td>(y = x^2 - 9)</td>
<td>(y = -x^2 + 4)</td>
<td>(y = -x^2 + 1)</td>
</tr>
<tr>
<td>18</td>
<td>(2x + 3y = 6)</td>
<td>(5x + 2y = 10)</td>
<td>(9x^2 + 4y = 36)</td>
<td>(4x^2 + y = 4)</td>
</tr>
</tbody>
</table>

**In Problems 29–38, plot each point. Then plot the point that is symmetric to it with respect to (a) the \(x\)-axis; (b) the \(y\)-axis; (c) the origin.**

<table>
<thead>
<tr>
<th></th>
<th>(3, 4)</th>
<th>(5, 3)</th>
<th>((-2, 1))</th>
<th>((-4, \ -2))</th>
<th>((-5, \ -2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>29</td>
<td>((-1, \ -1))</td>
<td>((-3, \ -4))</td>
<td>((4, 0))</td>
<td>((0, \ -3))</td>
<td>((-3, 0))</td>
</tr>
</tbody>
</table>
In Problems 39–50, the graph of an equation is given. (a) Find the intercepts. (b) Indicate whether the graph is symmetric with respect to the x-axis, the y-axis, or the origin.

39. $y = 3x - 9$
40. $y = x^2 - 4$
41. $y = \sqrt{x}$
42. $y = -x^2 + 4$

43. $y = \frac{3x}{x^2 + 9}$
44. $y = \frac{x^2 - 4}{2x}$
45. $y = \frac{-x^3}{x^2 - 9}$
46. $y = \frac{x^4 + 1}{2x^5}$

47. $y = 0.4$
48. $y = 0.2$
49. $y = 0.0$
50. $y = 0.0$

In Problems 51–54, draw a complete graph so that it has the type of symmetry indicated.

51. y-axis
52. x-axis
53. Origin
54. y-axis

55. $y^2 = x + 4$
56. $y^2 = x + 9$
57. $y = \sqrt{x}$
58. $y = \sqrt{x}$

59. $x^2 + y - 9 = 0$
60. $x^2 - y - 4 = 0$
61. $9x^2 + 4y^2 = 36$
62. $4x^2 + y^2 = 4$

63. $y = x^3 - 27$
64. $y = x^4 - 1$
65. $y = x^2 - 3x - 4$
66. $y = x^2 + 4$

67. $y = \frac{3x}{x^2 + 9}$
68. $y = \frac{x^2 - 4}{2x}$
69. $y = \frac{-x^3}{x^2 - 9}$
70. $y = \frac{x^4 + 1}{2x^5}$

In Problems 55–70, list the intercepts and test for symmetry.

71. $y = x^3$
72. $x = y^2$
73. $y = \sqrt{x}$
74. $y = \frac{1}{x}$

75. If $(3, b)$ is a point on the graph of $y = 4x + 1$, what is $b$?
76. If $(-2, b)$ is a point on the graph of $2x + 3y = 2$, what is $b$?
77. If $(a, 4)$ is a point on the graph of $y = x^2 + 3x$, what is $a$?
78. If $(a, -5)$ is a point on the graph of $y = x^2 + 6x$, what is $a$?
Applications and Extensions

79. Given that the point (1, 2) is on the graph of an equation that is symmetric with respect to the origin, what other point is on the graph?

80. If the graph of an equation is symmetric with respect to the y-axis and 6 is an x-intercept of this graph, name another x-intercept.

81. If the graph of an equation is symmetric with respect to the origin and -4 is an x-intercept of this graph, name another x-intercept.

82. If the graph of an equation is symmetric with respect to the x-axis and 2 is a y-intercept, name another y-intercept.

83. Microphones In studios and on stages, cardioid microphones are often preferred for the richness they add to voices and for their ability to reduce the level of sound from the sides and rear of the microphone. Suppose one such cardioid pattern is given by the equation

\[ 1x^2 + y^2 - x^2 = x^2 + y^2. \]

(a) Find the intercepts of the graph of the equation.
(b) Test for symmetry with respect to the x-axis, y-axis, and origin.

Source: www.notaviva.com

Discussion and Writing

In Problem 85, you may use a graphing utility, but it is not required.

85. (a) Graph \( y = \sqrt{x}, y = x, y = |x|, \) and \( y = (\sqrt{x})^2, \) noting which graphs are the same.
(b) Explain why the graphs of \( y = \sqrt{x^2} \) and \( y = |x| \) are the same.
(c) Explain why the graphs of \( y = x \) and \( y = (\sqrt{x})^2 \) are not the same.
(d) Explain why the graphs of \( y = \sqrt{x^2} \) and \( y = x \) are not the same.

86. Explain what is meant by a complete graph.

87. Draw a graph of an equation that contains two x-intercepts; at one the graph crosses the x-axis, and at the other the graph touches the x-axis.

'Are You Prepared?' Answers

1. \{-6\} 2. \{-3, 3\}

2.3 Lines

OBJECTIVES

1 Calculate and Interpret the Slope of a Line (p. 174)
2 Graph Lines Given a Point and the Slope (p. 176)
3 Find the Equation of a Vertical Line (p. 177)
4 Use the Point-Slope Form of a Line; Identify Horizontal Lines (p. 178)
5 Find the Equation of a Line Given Two Points (p. 179)
6 Write the Equation of a Line in Slope-Intercept Form (p. 179)
7 Identify the Slope and y-Intercept of a Line from Its Equation (p. 180)
8 Graph Lines Written in General Form Using Intercepts (p. 181)
9 Find Equations of Parallel Lines (p. 182)
10 Find Equations of Perpendicular Lines (p. 183)

Solar Energy The solar electric generating systems at Kramer Junction, California, use parabolic troughs to heat a heat-transfer fluid to a high temperature. This fluid is used to generate steam that drives a power conversion system to produce electricity. For troughs 7.5 feet wide, an equation for the cross-section is \( 16y^2 = 120x - 225. \)

(a) Find the intercepts of the graph of the equation.
(b) Test for symmetry with respect to the x-axis, y-axis, and origin.

Source: U.S. Department of Energy
In this section we study a certain type of equation that contains two variables, called a linear equation, and its graph, a line.

1. **Calculate and Interpret the Slope of a Line**

Consider the staircase illustrated in Figure 26. Each step contains exactly the same horizontal run and the same vertical rise. The ratio of the rise to the run, called the slope, is a numerical measure of the steepness of the staircase. For example, if the run is increased and the rise remains the same, the staircase becomes less steep. If the run is kept the same, but the rise is increased, the staircase becomes more steep. This important characteristic of a line is best defined using rectangular coordinates.

**DEFINITION**

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two distinct points. If $x_1 \neq x_2$, the slope $m$ of the nonvertical line $L$ containing $P$ and $Q$ is defined by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad x_1 \neq x_2$$

(1)

If $x_1 = x_2$, $L$ is a vertical line and the slope $m$ of $L$ is undefined (since this results in division by 0).

Figure 27(a) provides an illustration of the slope of a nonvertical line; Figure 27(b) illustrates a vertical line.

As Figure 27(a) illustrates, the slope $m$ of a nonvertical line may be viewed as

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Rise}}{\text{Run}}$$

Two comments about computing the slope of a nonvertical line may prove helpful:

1. Any two distinct points on the line can be used to compute the slope of the line. (See Figure 28 for justification.)
2. The slope of a line may be computed from \( P = (x_1, y_1) \) to \( Q = (x_2, y_2) \) or from \( Q \) to \( P \) because

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}
\]

Finally, we can also express the slope \( m \) of a nonvertical line as

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \text{Change in } y \quad \text{Change in } x = \frac{\Delta y}{\Delta x}
\]

That is, the slope \( m \) of a nonvertical line \( L \) measures the amount that \( y \) changes as \( x \) changes from \( x_1 \) to \( x_2 \). The expression \( \frac{\Delta y}{\Delta x} \) is called the average rate of change of \( y \) with respect to \( x \).

Since any two distinct points can be used to compute the slope of a line, the average rate of change of a line is always the same number.

**EXAMPLE 1**

**Finding and Interpreting the Slope of a Line Given Two Points**

The slope \( m \) of the line containing the points \((1, 2)\) and \((5, -3)\) may be computed as

\[
m = \frac{-3 - 2}{5 - 1} = \frac{-5}{4} = -\frac{5}{4}
\]

or as \( m = \frac{2 - (-3)}{1 - 5} = \frac{5}{-4} = -\frac{5}{4} \).

For every 4-unit change in \( x \), \( y \) will change by \(-5 \) units. That is, if \( x \) increases by 4 units, then \( y \) will decrease by 5 units. The average rate of change of \( y \) with respect to \( x \) is \(-\frac{5}{4} \).

---

**EXAMPLE 2**

**Finding the Slopes of Various Lines Containing the Same Point \((2, 3)\)**

Compute the slopes of the lines \( L_1, L_2, L_3, \) and \( L_4 \) containing the following pairs of points. Graph all four lines on the same set of coordinate axes.

- \( L_1: \ P = (2, 3) \quad Q_1 = (-1, -2) \)
- \( L_2: \ P = (2, 3) \quad Q_2 = (3, -1) \)
- \( L_3: \ P = (2, 3) \quad Q_3 = (5, 3) \)
- \( L_4: \ P = (2, 3) \quad Q_4 = (2, 5) \)

Let \( m_1, m_2, m_3, \) and \( m_4 \) denote the slopes of the lines \( L_1, L_2, L_3, \) and \( L_4, \) respectively. Then

\[
m_1 = \frac{-2 - 3}{-1 - 2} = \frac{-5}{-3} = \frac{5}{3} \quad \text{A rise of 5 divided by a run of 3}
\]

\[
m_2 = \frac{-1 - 3}{3 - 2} = \frac{-4}{1} = -4
\]

\[
m_3 = \frac{3 - 3}{5 - 2} = \frac{0}{3} = 0
\]

\( m_4 \) is undefined because \( x_1 = x_2 = 2 \)

The graphs of these lines are given in Figure 29.
Figure 29 illustrates the following facts:

1. When the slope of a line is positive, the line slants upward from left to right \((L_1)\).
2. When the slope of a line is negative, the line slants downward from left to right \((L_2)\).
3. When the slope is 0, the line is horizontal \((L_3)\).
4. When the slope is undefined, the line is vertical \((L_4)\).

See Figure 30.

Figures 30 and 31 illustrate that the closer the line is to the vertical position, the greater the magnitude of the slope.

2 Graph Lines Given a Point and the Slope

The next example illustrates how the slope of a line can be used to graph the line.

EXAMPLE 3 Graphing a Line Given a Point and a Slope

Draw a graph of the line that contains the point \((3, 2)\) and has a slope of:

(a) \(\frac{3}{4}\)  
(b) \(-\frac{4}{5}\)

**Solution**

(a) \(\text{Slope} = \frac{\text{Rise}}{\text{Run}}\). The fact that the slope is \(\frac{3}{4}\) means that for every horizontal movement (run) of 4 units to the right there will be a vertical movement (rise) of 3 units. Look at Figure 32. If we start at the given point \((3, 2)\) and move 4 units to the right and 3 units up, we reach the point \((7, 5)\). By drawing the line through this point and the point \((3, 2)\), we have the graph.

See Figure 31.
(b) The fact that the slope is
\[ \frac{4}{5} = \frac{-4}{5} \]
means that for every horizontal movement of 5 units to the right there will be a corresponding vertical movement of 4 units (a downward movement). If we start at the given point (3, 2) and move 5 units to the right and then 4 units down, we arrive at the point \((8, -2)\). By drawing the line through these points, we have the graph. See Figure 33.

Alternatively, we can set
\[ \frac{4}{-5} = \frac{4}{-5} \]
so that for every horizontal movement of 5 units (a movement to the left) there will be a corresponding vertical movement of 4 units (upward). This approach brings us to the point \((-2, 6)\), which is also on the graph shown in Figure 33.

### 3 Find the Equation of a Vertical Line

Now that we have discussed the slope of a line, we are ready to derive equations of lines. As we shall see, there are several forms of the equation of a line. Let's start with an example.

**EXAMPLE 4**

**Graphing a Line**

Graph the equation: \(x = 3\)

**Solution**

To graph \(x = 3\), recall that we are looking for all points \((x, y)\) in the plane for which \(x = 3\). No matter what \(y\)-coordinate is used, the corresponding \(x\)-coordinate always equals 3. Consequently, the graph of the equation \(x = 3\) is a vertical line with \(x\)-intercept 3 and undefined slope. See Figure 34.

As suggested by Example 4, we have the following result:

**THEOREM**

**Equation of a Vertical Line**

A vertical line is given by an equation of the form

\[ x = a \]

where \(a\) is the \(x\)-intercept.
COMMENT  To graph an equation using a graphing utility, we need to express the equation in the form \( y = \{\text{expression in } x}\). But \( x = 3 \) cannot be put in this form. To overcome this, most graphing utilities have special commands for drawing vertical lines. DRAW, LINE, PLOT, and VERT are among the more common ones. Consult your manual to determine the correct methodology for your graphing utility.

4 Use the Point–Slope Form of a Line; Identify Horizontal Lines

Now let \( L \) be a nonvertical line with slope \( m \) and containing the point \((x_1, y_1)\). See Figure 35. For any other point \((x, y)\) on \( L \), we have

\[
m = \frac{y - y_1}{x - x_1} \quad \text{or} \quad y - y_1 = m(x - x_1)
\]

THEOREM  Point–Slope Form of an Equation of a Line

An equation of a nonvertical line with slope \( m \) that contains the point \((x_1, y_1)\) is

\[
y - y_1 = m(x - x_1)
\]

EXAMPLE 5  Using the Point–Slope Form of a Line

An equation of the line with slope 4 and containing the point \((1, 2)\) can be found by using the point–slope form with \( m = 4 \), \( x_1 = 1 \), and \( y_1 = 2 \).

\[
y - y_1 = m(x - x_1) \
\]

\[
y - 2 = 4(x - 1) \quad m = 4, \quad x_1 = 1, \quad y_1 = 2
\]

\[
y = 4x - 2 \quad \text{Solve for } y.
\]

See Figure 36 for the graph.

EXAMPLE 6  Finding the Equation of a Horizontal Line

Find an equation of the horizontal line containing the point \((3, 2)\).

Solution  Because all the \( y \)-values are equal on a horizontal line, the slope of a horizontal line is 0. To get an equation, we use the point–slope form with \( m = 0 \), \( x_1 = 3 \), and \( y_1 = 2 \).

\[
y - y_1 = m(x - x_1) \
\]

\[
y - 2 = 0 \cdot (x - 3) \quad m = 0, \quad x_1 = 3, \quad \text{and } \quad y_1 = 2
\]

\[
y = 2
\]

See Figure 37 for the graph.
As suggested by Example 6, we have the following result:

**THEOREM**

**Equation of a Horizontal Line**

A horizontal line is given by an equation of the form

\[ y = b \]

where \( b \) is the \( y \)-intercept.

5 **Find the Equation of a Line Given Two Points**

We use the slope formula and the point-slope form of a line to find the equation of a line given two points.

**EXAMPLE 7**

**Finding an Equation of a Line Given Two Points**

Find an equation of the line containing the points \((2, 3)\) and \((-4, 5)\). Graph the line.

**Solution**

We first compute the slope of the line.

\[ m = \frac{5 - 3}{-4 - 2} = \frac{2}{-6} = -\frac{1}{3} \]

We use the point \((2, 3)\) and the slope \( m = -\frac{1}{3} \) to get the point-slope form of the equation of the line.

\[ y - 3 = -\frac{1}{3}(x - 2) \]

See Figure 38 for the graph.

In the solution to Example 7, we could have used the other point, \((-4, 5)\), instead of the point \((2, 3)\). The equation that results, although it looks different, is equivalent to the equation that we obtained in the example. (Try it for yourself.)

6 **Write the Equation of a Line in Slope-Intercept Form**

Another useful equation of a line is obtained when the slope \( m \) and \( y \)-intercept \( b \) are known. In this event, we know both the slope \( m \) of the line and a point \((0, b)\) on the line; then we may use the point-slope form, equation (2), to obtain the following equation:

\[ y - b = m(x - 0) \quad \text{or} \quad y = mx + b \]

**THEOREM**

**Slope-Intercept Form of an Equation of a Line**

An equation of a line with slope \( m \) and \( y \)-intercept \( b \) is

\[ y = mx + b \] (3)

**Now Work**

**Problem 51 (Express Answer in Slope-Intercept Form)**
CHAPTER 2  Graphs

To see the role that the slope \( m \) plays, graph the following lines on the same screen.

See Figure 39. What do you conclude about the lines \( y = mx + 2 \)?

\[
\begin{align*}
Y_1 &= 2 \\
Y_2 &= x + 2 \\
Y_3 &= -x + 2 \\
Y_4 &= 3x + 2 \\
Y_5 &= -3x + 2
\end{align*}
\]

See Figure 39. What do you conclude about the lines \( y = mx + 2 \)?

To see the role of the \( y \)-intercept \( b \), graph the following lines on the same screen.

See Figure 40. What do you conclude about the lines \( y = 2x + b \)?

\[
\begin{align*}
Y_1 &= 2x \\
Y_2 &= 2x + 1 \\
Y_3 &= 2x - 1 \\
Y_4 &= 2x + 4 \\
Y_5 &= 2x - 4
\end{align*}
\]

See Figure 40. What do you conclude about the lines \( y = 2x + b \)?

**7 Identify the Slope and \( y \)-Intercept of a Line from Its Equation**

When the equation of a line is written in slope–intercept form, it is easy to find the slope \( m \) and \( y \)-intercept \( b \) of the line. For example, suppose that the equation of a line is

\[ y = -2x + 7 \]

Compare it to \( y = mx + b \).

\[ y = -2x + 7 \]

The slope of this line is \(-2\) and its \( y \)-intercept is 7.

**EXAMPLE 8 Finding the Slope and \( y \)-Intercept**

Find the slope \( m \) and \( y \)-intercept \( b \) of the equation \( 2x + 4y = 8 \). Graph the equation.

**Solution**

To obtain the slope and \( y \)-intercept, we write the equation in slope–intercept form by solving for \( y \).

\[
2x + 4y = 8 \\
4y = -2x + 8 \\
y = -\frac{1}{2}x + 2 \quad y = mx + b
\]

The coefficient of \( x \), \(-\frac{1}{2}\), is the slope, and the \( y \)-intercept is 2. We can graph the line using the fact that the \( y \)-intercept is 2 and the slope is \(-\frac{1}{2}\). Then, starting at the point \((0, 2)\), go to the right 2 units and then down 1 unit to the point \((2, 1)\). See Figure 41.

**New Work**  **Problem 71**

**New Work**  **Problem 77**
Graph Lines Written in General Form Using Intercepts

Refer to Example 8. The form of the equation of the line $2x + 4y = 8$, is called the general form.

**DEFINITION**

The equation of a line is in **general form** when it is written as

$$Ax + By = C$$ (4)

where $A$, $B$, and $C$ are real numbers and $A$ and $B$ are not both 0.

When we want to graph an equation that is written in general form, we can solve the equation for $y$ and write the equation in slope–intercept form as we did in Example 8. Another approach to graphing the equation would be to find its intercepts. Remember, the intercepts of the graph of an equation are the points where the graph crosses or touches a coordinate axis.

**EXAMPLE 9**

Graphing an Equation in General Form Using Its Intercepts

Graph the equation $2x + 4y = 8$ by finding its intercepts.

**Solution**

To obtain the $x$-intercept, let $y = 0$ in the equation and solve for $x$.

$$2x + 4y = 8$$

$$2x + 4(0) = 8 \quad \text{Let } y = 0.$$ 

$$2x = 8$$

$$x = 4 \quad \text{Divide both sides by 2.}$$

The $x$-intercept is 4 and the point $(4, 0)$ is on the graph of the equation.

To obtain the $y$-intercept, let $x = 0$ in the equation and solve for $y$.

$$2x + 4y = 8$$

$$2(0) + 4y = 8 \quad \text{Let } x = 0.$$ 

$$4y = 8$$

$$y = 2 \quad \text{Divide both sides by 4.}$$

The $y$-intercept is 2 and the point $(0, 2)$ is on the graph of the equation.

We plot the points $(4, 0)$ and $(0, 2)$ on a Cartesian plane and draw a line through the points. See Figure 42.

---

**Now Work Problem 91**

Every line has an equation that is equivalent to an equation written in general form. For example, a vertical line whose equation is

$$x = a$$

can be written in the general form

$$1 \cdot x + 0 \cdot y = a \quad A = 1, B = 0, C = a$$

A horizontal line whose equation is

$$y = b$$

can be written in the general form

$$0 \cdot x + 1 \cdot y = b \quad A = 0, B = 1, C = b$$

*Some books use the term **standard form**.
Lines that are neither vertical nor horizontal have general equations of the form

\[ Ax + By = C \quad A \neq 0 \text{ and } B \neq 0 \]

Because the equation of every line can be written in general form, any equation equivalent to equation (4) is called a linear equation.

### Find Equations of Parallel Lines

When two lines (in the plane) do not intersect (that is, they have no points in common), they are said to be parallel. Look at Figure 43. There we have drawn two parallel lines and have constructed two right triangles by drawing sides parallel to the coordinate axes. The right triangles are similar. (Do you see why? Two angles are equal.) Because the triangles are similar, the ratios of corresponding sides are equal.

This suggests the following result:

**THEOREM**

**Criterion for Parallel Lines**

Two nonvertical lines are parallel if and only if their slopes are equal and they have different \(y\)-intercepts.

The use of the words “if and only if” in the preceding theorem means that actually two statements are being made, one the converse of the other.

If two nonvertical lines are parallel, then their slopes are equal and they have different \(y\)-intercepts.

If two nonvertical lines have equal slopes and they have different \(y\)-intercepts, then they are parallel.

### EXAMPLE 10  Showing That Two Lines Are Parallel

Show that the lines given by the following equations are parallel:

\[ L_1: \ 2x + 3y = 6, \quad L_2: \ 4x + 6y = 0 \]

**Solution**

To determine whether these lines have equal slopes and different \(y\)-intercepts, we write each equation in slope–intercept form:

\[ L_1: \ 2x + 3y = 6 \quad \Rightarrow \quad 3y = -2x + 6 \quad \Rightarrow \quad y = -\frac{2}{3}x + 2 \]

\[ L_2: \ 4x + 6y = 0 \quad \Rightarrow \quad 6y = -4x \quad \Rightarrow \quad y = -\frac{2}{3}x \]

Slope \(= -\frac{2}{3}; \) \(y\)-intercept \(= 2\) \quad Slope \(= -\frac{2}{3}; \) \(y\)-intercept \(= 0\)

Because these lines have the same slope, \(-\frac{2}{3}\), but different \(y\)-intercepts, the lines are parallel. See Figure 44.

### EXAMPLE 11  Finding a Line That Is Parallel to a Given Line

Find an equation for the line that contains the point \((2, -3)\) and is parallel to the line \(2x + y = 6\).
**Solution**

Since the two lines are to be parallel, the slope of the line that we seek equals the slope of the line $2x + y = 6$. We begin by writing the equation of the line $2x + y = 6$ in slope–intercept form.

$$2x + y = 6$$
$$y = -2x + 6$$

The slope is $-2$. Since the line that we seek also has slope $-2$ and contains the point $(2, -3)$, we use the point–slope form to obtain its equation.

$$y - y_1 = m(x - x_1)$$  \hspace{1cm} \text{Point–slope form}  \\
y - (-3) = -2(x - 2)$$

$$y + 3 = -2x + 4$$  \hspace{1cm} \text{Simplify}  \\
y = -2x + 1$$  \hspace{1cm} \text{Slope–intercept form}  \\
$2x + y = 6$  \hspace{1cm} \text{General form}

This line is parallel to the line $2x + y = 6$ and contains the point $(2, -3)$. See Figure 45.

---

### Find Equations of Perpendicular Lines

When two lines intersect at a right angle (90°), they are said to be **perpendicular**. See Figure 46.

The following result gives a condition, in terms of their slopes, for two lines to be perpendicular.

**THEOREM**

**Criterion for Perpendicular Lines**

Two nonvertical lines are perpendicular if and only if the product of their slopes is $-1$.

Here we shall prove the “only if” part of the statement:

If two nonvertical lines are perpendicular, then the product of their slopes is $-1$.

In Problem 128 you are asked to prove the “if” part of the theorem; that is:

If two nonvertical lines have slopes whose product is $-1$, then the lines are perpendicular.
Proof Let $m_1$ and $m_2$ denote the slopes of the two lines. There is no loss in generality (that is, neither the angle nor the slopes are affected) if we situate the lines so that they meet at the origin. See Figure 47. The point $A = (1, m_2)$ is on the line having slope $m_2$, and the point $B = (1, m_1)$ is on the line having slope $m_1$. (Do you see why this must be true?)

Suppose that the lines are perpendicular. Then triangle $OAB$ is a right triangle. As a result of the Pythagorean Theorem, it follows that

$$[d(O, A)]^2 + [d(O, B)]^2 = [d(A, B)]^2$$

(5)

Using the distance formula, we can write the squares of these distances as

$$[d(O, A)]^2 = (1 - 0)^2 + (m_2 - 0)^2 = 1 + m_2^2$$
$$[d(O, B)]^2 = (1 - 0)^2 + (m_1 - 0)^2 = 1 + m_1^2$$
$$[d(A, B)]^2 = (1 - 1)^2 + (m_2 - m_1)^2 = m_2^2 - 2m_1m_2 + m_1^2$$

Using these facts in equation (5), we get

$$(1 + m_2^2) + (1 + m_1^2) = m_2^2 - 2m_1m_2 + m_1^2$$

which, upon simplification, can be written as

$$m_1m_2 = -1$$

If the lines are perpendicular, the product of their slopes is $-1$.

You may find it easier to remember the condition for two nonvertical lines to be perpendicular by observing that the equality means that $m_1$ and $m_2$ are negative reciprocals of each other; that is, either $m_1 = -\frac{1}{m_2}$ or $m_2 = -\frac{1}{m_1}$.

**Example 12** Finding the Slope of a Line Perpendicular to Another Line

If a line has slope $\frac{3}{2}$, any line having slope $-\frac{2}{3}$ is perpendicular to it.

**Example 13** Finding the Equation of a Line Perpendicular to a Given Line

Find an equation of the line that contains the point $(1, -2)$ and is perpendicular to the line $x + 3y = 6$. Graph the two lines.

**Solution** We first write the equation of the given line in slope–intercept form to find its slope.

$$x + 3y = 6$$
$$3y = -x + 6 \quad \text{Proceed to solve for } y,$n
$$y = -\frac{1}{3}x + 2 \quad \text{Place in the form } y = mx + b.$$n

The given line has slope $-\frac{1}{3}$. Any line perpendicular to this line will have slope 3. Because we require the point $(1, -2)$ to be on this line with slope 3, we use the point–slope form of the equation of a line.

$$y - y_1 = m(x - x_1) \quad \text{Point–slope form}$$
$$y - (-2) = 3(x - 1) \quad m = 3, x_1 = 1, y_1 = -2$$
To obtain other forms of the equation, we proceed as follows:

\[ y + 2 = 3(x - 1) \]
\[ y + 2 = 3x - 3 \quad \text{Simplify} \]
\[ y = 3x - 5 \quad \text{Slope–intercept form} \]
\[ 3x - y = 5 \quad \text{General form} \]

Figure 48 shows the graphs.

**2.3 Assess Your Understanding**

**Concepts and Vocabulary**

1. The slope of a vertical line is _________; the slope of a horizontal line is _________.
2. For the line \( 2x + 3y = 6 \), the x-intercept is _________ and the y-intercept is _________.
3. A horizontal line is given by an equation of the form _________, where \( b \) is the _________.
4. **True or False** Vertical lines have an undefined slope.
5. **True or False** The slope of the line \( 2y = 3x + 5 \) is 3.
6. **True or False** The point \((1, 2)\) is on the line \( 2x + y = 4 \).
7. Two nonvertical lines have slopes \( m_1 \) and \( m_2 \), respectively. The lines are parallel if _________ and the_______ are unequal; the lines are perpendicular if _________.
8. The lines \( y = 2x + 3 \) and \( y = ax + 5 \) are parallel if \( a = _________ \).
9. The lines \( y = 2x - 1 \) and \( y = ax + 2 \) are perpendicular if \( a = _________ \).
10. **True or False** Perpendicular lines have slopes that are reciprocals of one another.

**Skill Building**

In Problems 11–14, (a) find the slope of the line and (b) interpret the slope.

11. \( y \)
    \[ (0, 0) \quad (2, 1) \]
    \[ x \]
    \[ -2 \quad -1 \quad 1 \quad 2 \]
12. \( y \)
    \[ (-2, 1) \quad (0, 0) \]
    \[ x \]
    \[ -2 \quad -1 \quad 1 \quad 2 \]
13. \( y \)
    \[ (2, 1) \quad (-2, 2) \]
    \[ x \]
    \[ -2 \quad -1 \quad 1 \quad 2 \]
14. \( y \)
    \[ (-1, 1) \quad (2, 2) \]
    \[ x \]
    \[ -2 \quad -1 \quad 1 \quad 2 \]

In Problems 15–22, plot each pair of points and determine the slope of the line containing them. Graph the line.

15. \((2, 3); (4, 0)\)
16. \((4, 2); (3, 4)\)
17. \((-2, 3); (2, 1)\)
18. \((-1, 1); (2, 3)\)
19. \((-3, -1); (2, -1)\)
20. \((4, 2); (-5, 2)\)
21. \((-1, 2); (-1, -2)\)
22. \((2, 0); (2, 2)\)

In Problems 23–30, graph the line containing the point \( P \) and having slope \( m \).

23. \( P = (1, 2); m = 3 \)
24. \( P = (2, 1); m = 4 \)
25. \( P = (2, 4); m = -\frac{3}{4} \)
26. \( P = (1, 3); m = -\frac{2}{5} \)
27. \( P = (-1, 3); m = 0 \)
28. \( P = (2, -4); m = 0 \)
29. \( P = (0, 3); \text{slope undefined} \)
30. \( P = (-2, 0); \text{slope undefined} \)

In Problems 31–36, the slope and a point on a line are given. Use this information to locate three additional points on the line. Answers may vary.

[H] **[Hint: It is not necessary to find the equation of the line. See Example 3.]**

31. Slope 4; point \((1, 2)\)
32. Slope 2; point \((-2, 3)\)
33. Slope \(-\frac{3}{2}\); point \((2, -4)\)
34. Slope \(\frac{4}{3}\); point \((-3, 2)\)
35. Slope \(-2\); point \((-2, -3)\)
36. Slope \(-1\); point \((4, 1)\)
In Problems 37–44, find an equation of the line L.

37.\[ y = 2x + 3 \]
38.\[ y = -3x + 4 \]
39.\[ \frac{1}{2}y = x - 1 \]
40.\[ \frac{1}{3}x + y = 2 \]
41.\[ y = 2x \]
42.\[ y = -x \]
43.\[ y = 2x \]
44.\[ y = -x \]

$L$ is parallel to $y = 2x$
$L$ is parallel to $y = -x$
$L$ is perpendicular to $y = 2x$
$L$ is perpendicular to $y = -x$

In Problems 45–70, find an equation for the line with the given properties. Express your answer using either the general form or the slope–intercept form of the equation of a line, whichever you prefer.

45. Slope = 3; containing the point $(-2, 3)$
46. Slope = 2; containing the point $(4, -3)$
47. Slope = $-\frac{2}{3}$; containing the point $(1, -1)$
48. Slope = $\frac{1}{2}$; containing the point $(3, 1)$
49. Containing the points $(1, 3)$ and $(-1, 2)$
50. Containing the points $(-3, 4)$ and $(2, 5)$
51. Slope = $-3$; $y$-intercept = 3
52. Slope = $-2$; $y$-intercept = $-2$
53. $x$-intercept = 2; $y$-intercept = $-1$
54. $x$-intercept = $-4$; $y$-intercept = 4
55. Slope undefined; containing the point $(2, 4)$
56. Slope undefined; containing the point $(3, 8)$
57. Horizontal; containing the point $(-3, 2)$
58. Vertical; containing the point $(4, -5)$
59. Parallel to the line $y = 2x$; containing the point $(-1, 2)$
60. Parallel to the line $y = -3x$; containing the point $(-1, 2)$
61. Parallel to the line $2x - y = -2$; containing the point $(0, 0)$
62. Parallel to the line $x - 2y = -5$; containing the point $(0, 0)$
63. Parallel to the line $x = 5$; containing the point $(4, 2)$
64. Parallel to the line $y = 5$; containing the point $(4, 2)$
65. Perpendicular to the line $y = \frac{1}{2}x + 4$; containing the point $(1, -2)$
66. Perpendicular to the line $y = 2x - 3$; containing the point $(1, -2)$
67. Perpendicular to the line $2x + y = 2$; containing the point $(-3, 0)$
68. Perpendicular to the line $x - 2y = -5$; containing the point $(0, 4)$
69. Perpendicular to the line $x = 8$; containing the point $(3, 4)$
70. Perpendicular to the line $y = 8$; containing the point $(3, 4)$

In Problems 71–90, find the slope and $y$-intercept of each line. Graph the line.

71. $y = 2x + 3$
72. $y = -3x + 4$
73. $\frac{1}{2}y = x - 1$
74. $\frac{1}{3}x + y = 2$
75. $y = \frac{1}{2}x + 2$
76. $y = 2x + \frac{1}{2}$
77. $x + 2y = 4$
78. $-x + 3y = 6$
79. $2x - 3y = 6$
80. $3x + 2y = 6$
81. $x + y = 1$
82. $x - y = 2$
83. $x = -4$
84. $y = -1$
85. $y = 5$
86. $x = 2$
87. $y - x = 0$
88. $x + y = 0$
89. $2y - 3x = 0$
90. $3x + 2y = 0$
In Problems 91–100, (a) find the intercepts of the graph of each equation and (b) graph the equation.

91. \(2x + 3y = 6\)  
92. \(3x - 2y = 6\)  
93. \(-4x + 5y = 40\)

94. \(6x - 4y = 24\)  
95. \(7x + 2y = 21\)  
96. \(5x + 3y = 18\)

97. \(\frac{1}{2}x + \frac{1}{3}y = 1\)  
98. \(x - \frac{2}{3}y = 4\)  
99. \(0.2x - 0.5y = 1\)  
100. \(-0.3x + 0.4y = 1.2\)

101. Find an equation of the x-axis.

102. Find an equation of the y-axis.

In Problems 103–106, the equations of two lines are given. Determine if the lines are parallel, perpendicular, or neither.

103. \(y = 2x - 3\)  
104. \(y = \frac{1}{2}x - 3\)  
105. \(y = 4x + 5\)  
106. \(y = -2x + 3\)

107. \(y = 2x + 4\)  
108. \(y = -2x + 4\)

In Problems 107–110, write an equation of each line. Express your answer using either the general form or the slope–intercept form of the equation of a line, whichever you prefer.

Applications and Extensions

111. Geometry Use slopes to show that the triangle whose vertices are \((-2, 5), (1, 3),\) and \((-1, 0)\) is a right triangle.

112. Geometry Use slopes to show that the quadrilateral whose vertices are \((1, -1), (4, 1), (2, 2),\) and \((5, 4)\) is a parallelogram.

113. Geometry Use slopes to show that the quadrilateral whose vertices are \((-1, 0), (2, 3), (1, -2),\) and \((4, 1)\) is a rectangle.

114. Geometry Use slopes and the distance formula to show that the quadrilateral whose vertices are \((0, 0), (1, 3), (4, 2),\) and \((3, -1)\) is a square.

115. Truck Rentals A truck rental company rents a moving truck for one day by charging $29 plus $0.20 per mile. Write a linear equation that relates the cost \(C\), in dollars, of renting the truck to the number \(x\) of miles driven. What is the cost of renting the truck if the truck is driven 110 miles? 230 miles?

116. Cost Equation The fixed costs of operating a business are the costs incurred regardless of the level of production. Fixed costs include rent, fixed salaries, and costs of leasing machinery. The variable costs of operating a business are the costs that change with the level of output. Variable costs include raw materials, hourly wages, and electricity. Suppose that a manufacturer of jeans has fixed daily costs of $500 and variable costs of $8 for each pair of jeans manufactured. Write a linear equation that relates the daily cost \(C\), in dollars, of manufacturing the jeans to the number \(x\) of jeans manufactured. What is the cost of manufacturing 400 pairs of jeans? 740 pairs?

117. Cost of Sunday Home Delivery The cost to the Chicago Tribune for Sunday home delivery is approximately $0.53 per newspaper with fixed costs of $1,070,000. Write a linear equation that relates the cost \(C\), in dollars, to the number \(x\) of kilowatt-hours used in a month, \(0 \leq x \leq 400\).

(a) Write a linear equation that relates the monthly charge \(C\), in dollars, to the number \(x\) of kilowatt-hours used in a month, \(0 \leq x \leq 400\).

(b) Graph this equation.
(c) Which of the following equations might have the graph shown? (More than one answer is possible.)

129. Which of the following equations might have the graph shown? (More than one answer is possible.)

(a) $2x + 3y = 6$
(b) $-2x + 3y = 6$
(c) $3x - 4y = -12$
(d) $x - y = 1$
(e) $x - y = -1$
(f) $y = 3x - 5$
(g) $y = 2x + 3$
(h) $y = -3x + 3$

(c) Design requirements stipulate that the maximum run be 30 feet and that the maximum slope be a drop of 1 inch for each 12 inches of run. Will this ramp meet the requirements? Explain.

(d) What slopes could be used to obtain the 30-inch rise and still meet design requirements?

Source: Commonwealth Edison Company, April, 2006.

120. Electricity Rates in Florida Florida Power & Light Company supplies electricity to residential customers for a monthly customer charge of $5.17 plus 10.07 cents per kilowatt-hour for up to 1000 kilowatt-hours.

(a) Write a linear equation that relates the monthly charge $C$, in dollars, to the number $x$ of kilowatt-hours used in a month, $0 \leq x \leq 1000$.

(b) Graph this equation.

(c) What is the monthly charge for using 200 kilowatt-hours?

(d) What is the monthly charge for using 500 kilowatt-hours?

(e) Interpret the slope of the line.


121. Measuring Temperature The relationship between Celsius (°C) and Fahrenheit (°F) degrees of measuring temperature is linear. Find a linear equation relating °C and °F if 0°C corresponds to 32°F and 100°C corresponds to 212°F. Use the equation to find the Celsius measure of 70°F.

(a) Write a linear equation relating °F and °C.

(b) Graph this equation.

(c) What is the monthly charge for using 200 kilowatt-hours?

(d) What is the monthly charge for using 500 kilowatt-hours?

(e) Interpret the slope of the line.

Source: www.drugfree.org

122. Measuring Temperature The Kelvin (K) scale for measuring temperature is obtained by adding 273 to the Celsius temperature.

(a) Write a linear equation relating K and °C.

(b) What is the monthly charge for using 200 kilowatt-hours?

(c) What is the monthly charge for using 500 kilowatt-hours?

(d) What slopes could be used to obtain the 30-inch rise and still meet design requirements?

Source: www.adaptiveaccess.com/wood_ramps.php

123. Access Ramp A wooden access ramp is being built to reach a platform that sits 30 inches above the floor. The ramp drops 2 inches for every 25-inch run.

(a) Write a linear equation that relates the height $y$ above the floor to the horizontal distance $x$ from the platform.

(b) Find and interpret the $x$-intercept of the graph of your equation.

124. Cigarette Use A study by the Partnership for a Drug-Free America indicated that, in 1998, 42% of teens in grades 7 through 12 had recently used cigarettes. A similar study in 2005 indicated that 22% of such teens had recently used cigarettes.

(a) Write a linear equation that relates the percent of teens $K$ and °C of money it spends on advertising. If it spends $40,000 on advertising, then 100,000 boxes of cereal will be sold, and if it spends $60,000, then 200,000 boxes will be sold.

(b) How much advertising is needed to sell 300,000 boxes of cereal?

(c) Interpret the slope.

(d) What is the monthly charge for using 300 kilowatt-hours?

Source: www.drugfree.org

125. Product Promotion A cereal company finds that the number of people who will buy one of its products in the first month that it is introduced is linearly related to the amount of money it spends on advertising. If it spends $40,000 on advertising, then 100,000 boxes of cereal will be sold, and if it spends $60,000, then 200,000 boxes will be sold.

(a) Write a linear equation that relates the amount $A$ spent on advertising to the number $x$ of boxes the company aims to sell.

(b) What is the monthly charge for using 300 kilowatt-hours?

(c) Do the intercepts have any meaningful interpretation?

(d) Use your equation to predict the percent for the year 2019. Is this result reasonable?

126. Show that the line containing the points $(a, b)$ and $(b, a)$, $a \neq b$, is perpendicular to the line $y = x$. Also show that the midpoint of $(a, b)$ and $(b, a)$ lies on the line $y = x$.

127. The equation $2x - y = C$ defines a family of lines, one line for each value of $C$. On one set of coordinate axes, graph the members of the family when $C = -4$, $C = 0$, and $C = 2$. Can you draw a conclusion from the graph about each member of the family?

128. Prove that if two nonvertical lines have slopes whose product is $-1$ then the lines are perpendicular. [Hint: Refer to Figure 47 and use the converse of the Pythagorean Theorem.]
131. The figure shows the graph of two parallel lines. Which of the following pairs of equations might have such a graph?

(a) \( x - 2y = 3 \)
\( x + 2y = 7 \)
(b) \( x + y = 2 \)
\( x + y = -1 \)
(c) \( x - y = -2 \)
\( x - y = 1 \)
(d) \( x - y = -2 \)
\( 2x - 2y = -4 \)
(e) \( x + 2y = 2 \)
\( x + 2y = -1 \)

132. The figure shows the graph of two perpendicular lines. Which of the following pairs of equations might have such a graph?

(a) \( y - 2x = 2 \)
\( y + 2x = -1 \)
(b) \( y - 2x = 0 \)
\( 2y + x = 0 \)
(c) \( 2y - x = 2 \)
\( 2y + x = -2 \)
(d) \( y - 2x = 2 \)
\( x + 2y = -1 \)
(e) \( 2x + y = -2 \)
\( 2y + x = -2 \)

133. \( m \) is for Slope The accepted symbol used to denote the slope of a line is the letter \( m \). Investigate the origin of this symbolism. Begin by consulting a French dictionary and looking up the French word monter. Write a brief essay on your findings.

134. Grade of a Road The term grade is used to describe the inclination of a road. How does this term relate to the notion of slope of a line? Is a 4% grade very steep? Investigate the grades of some mountainous roads and determine their slopes. Write a brief essay on your findings.

135. Carpentry Carpenters use the term pitch to describe the steepness of staircases and roofs. How does pitch relate to slope? Investigate typical pitches used for stairs and for roofs. Write a brief essay on your findings.

136. Can the equation of every line be written in slope-intercept form? Why?

137. Does every line have exactly one \( x \)-intercept and one \( y \)-intercept? Are there any lines that have no intercepts?

138. What can you say about two lines that have equal slopes and equal \( y \)-intercepts?

139. What can you say about two lines with the same \( x \)-intercept and the same \( y \)-intercept? Assume that the \( x \)-intercept is not 0.

140. If two distinct lines have the same slope, but different \( x \)-intercepts, can they have the same \( y \)-intercept?

141. If two distinct lines have the same \( y \)-intercept, but different slopes, can they have the same \( x \)-intercept?

142. Which form of the equation of a line do you prefer to use? Justify your position with an example that shows that your choice is better than another. Have reasons.

### 2.4 Circles

**PREPARING FOR THIS SECTION**  
Before getting started, review the following:

- Completing the Square (Section 1.2, pp. 99–100)
- Square Root Method (Section 1.2, pp. 98–99)

Now Work the "Are You Prepared?" problems on page 193.

**OBJECTIVES**

1. Write the Standard Form of the Equation of a Circle (p. 189)
2. Graph a Circle (p. 191)
3. Work with the General Form of the Equation of a Circle (p. 192)

**Write the Standard Form of the Equation of a Circle**

One advantage of a coordinate system is that it enables us to translate a geometric statement into an algebraic statement, and vice versa. Consider, for example, the following geometric statement that defines a circle.

**Definition**

A circle is a set of points in the \( xy \)-plane that are a fixed distance \( r \) from a fixed point \((h, k)\). The fixed distance \( r \) is called the radius, and the fixed point \((h, k)\) is called the center of the circle.
Figure 49 shows the graph of a circle. To find the equation, we let \((x, y)\) represent the coordinates of any point on a circle with radius \(r\) and center \((h, k)\). Then the distance between the points \((x, y)\) and \((h, k)\) must always equal \(r\). That is, by the distance formula
\[
\sqrt{(x - h)^2 + (y - k)^2} = r
\]
or, equivalently,
\[
(x - h)^2 + (y - k)^2 = r^2
\]

**DEFINITION**

The **standard form of an equation of a circle** with radius \(r\) and center \((h, k)\) is
\[
(x - h)^2 + (y - k)^2 = r^2 \quad (1)
\]

**THEOREM**

The standard form of an equation of a circle of radius \(r\) with center at the origin \((0, 0)\) is
\[
x^2 + y^2 = r^2 \quad (2)
\]

**DEFINITION**

If the radius \(r = 1\), the circle whose center is at the origin is called the **unit circle** and has the equation
\[
x^2 + y^2 = 1 \quad (3)
\]

See Figure 50. Notice that the graph of the unit circle is symmetric with respect to the \(x\)-axis, the \(y\)-axis, and the origin.

**EXAMPLE 1**

**Writing the Standard Form of the Equation of a Circle**

Write the standard form of the equation of the circle with radius 5 and center \((-3, 6)\).

**Solution**

Using the form of equation (1) and substituting the values \(r = 5\), \(h = -3\), and \(k = 6\), we have
\[
(x - h)^2 + (y - k)^2 = r^2
\]
\[
(x + 3)^2 + (y - 6)^2 = 25
\]
Graph a Circle

The graph of any equation in the form of equation (1) is that of a circle with radius $r$ and center $(h, k)$.

**Graphing a Circle**

Graph the equation: $x^2 + y^2 = 16$

**Solution**

Since the equation is in the form of equation (1), its graph is a circle. To graph the equation, we first compare the given equation to the standard form of the equation of a circle. The comparison yields information about the circle.

We see that $h = -3$, $k = 2$, and $r = 4$. The circle has center $(-3, 2)$ and a radius of 4 units. To graph this circle, we first plot the center $(-3, 2)$. Since the radius is 4, we can locate four points on the circle by plotting points 4 units to the left, to the right, up, and down from the center. These four points can then be used as guides to obtain the graph. See Figure 51.

**Finding the Intercepts of a Circle**

For the circle $(x + 3)^2 + (y - 2)^2 = 16$, find the intercepts, if any, of its graph.

**Solution**

This is the equation discussed and graphed in Example 2. To find the $x$-intercepts, if any, let $y = 0$. Then

$(x + 3)^2 + (y - 2)^2 = 16$
$(x + 3)^2 + (0 - 2)^2 = 16$
$(x + 3)^2 + 4 = 16$
$(x + 3)^2 = 12$
$x + 3 = \pm \sqrt{12}$
$x = -3 \pm 2\sqrt{3}$ \hspace{1cm} (Apply the Square Root Method.)

The $x$-intercepts are $-3 - 2\sqrt{3} \approx -6.46$ and $-3 + 2\sqrt{3} \approx 0.46$.

To find the $y$-intercepts, if any, we let $x = 0$. Then

$(x + 3)^2 + (y - 2)^2 = 16$
$(0 + 3)^2 + (y - 2)^2 = 16$
$9 + (y - 2)^2 = 16$
$(y - 2)^2 = 7$
$y - 2 = \pm \sqrt{7}$
$y = 2 \pm \sqrt{7}$

The $y$-intercepts are $2 - \sqrt{7} \approx -0.65$ and $2 + \sqrt{7} \approx 4.65$.

Look back at Figure 51 to verify the approximate locations of the intercepts.
Work with the General Form of the Equation of a Circle

If we eliminate the parentheses from the standard form of the equation of the circle given in Example 2, we get

\[(x + 3)^2 + (y - 2)^2 = 16\]
\[x^2 + 6x + 9 + y^2 - 4y + 4 = 16\]

which we find, upon simplifying, is equivalent to

\[x^2 + y^2 + 6x - 4y - 3 = 0\]  \hspace{1cm} (2)

It can be shown that any equation of the form

\[x^2 + y^2 + ax + by + c = 0\]

has a graph that is a circle, or a point, or has no graph at all. For example, the graph of the equation \(x^2 + y^2 = 0\) is the single point \((0, 0)\). The equation \(x^2 + y^2 + 5 = 0\), or \(x^2 + y^2 = -5\), has no graph, because sums of squares of real numbers are never negative.

**DEFINITION**

When its graph is a circle, the equation

\[x^2 + y^2 + ax + by + c = 0\]

is referred to as the **general form of the equation of a circle**.

**New Work**  **Problem 13**

If an equation of a circle is in the general form, we use the method of completing the square to put the equation in standard form so that we can identify its center and radius.

**Example 4  **Graphing a Circle Whose Equation Is in General Form

Graph the equation \(x^2 + y^2 + 4x - 6y + 12 = 0\)

**Solution**

We complete the square in both \(x\) and \(y\) to put the equation in standard form. Group the expression involving \(x\), group the expression involving \(y\), and put the constant on the right side of the equation. The result is

\[(x^2 + 4x) + (y^2 - 6y) = -12\]

Next, complete the square of each expression in parentheses. Remember that any number added on the left side of the equation must also be added on the right.

\[\left(\frac{x + 2}{2}\right)^2 = 4 \quad \left(\frac{y - 3}{2}\right)^2 = 9\]

\[\left(x + 2\right)^2 + \left(y - 3\right)^2 = \text{Factor.}\]

We recognize this equation as the standard form of the equation of a circle with radius 1 and center \((-2, 3)\).

To graph the equation use the center \((-2, 3)\) and the radius 1. See Figure 52.
In Problems 21–34, (a) find the center and radius of each circle; (b) graph each circle; (c) find the intercepts, if any.

21. \( x^2 + y^2 = 4 \)
22. \( x^2 + (y - 1)^2 = 1 \)
23. \( 2(x - 3)^2 + 2y^2 = 8 \)
24. \( 3(x + 1)^2 + 3(y - 1)^2 = 6 \)
25. \( x^2 + y^2 - 2x - 4y - 4 = 0 \)
26. \( x^2 + y^2 + 4x + 2y - 20 = 0 \)

Using a Graphing Utility to Graph a Circle

Graph the equation: \( x^2 + y^2 = 4 \)

This is the equation of a circle with center at the origin and radius 2. To graph this equation, we must first solve for \( y \).

\[
\begin{align*}
  x^2 + y^2 &= 4 \\
  y^2 &= 4 - x^2 \\
  y &= \pm \sqrt{4 - x^2}
\end{align*}
\]

There are two equations to graph: first, we graph \( y_1 = \sqrt{4 - x^2} \) and then \( y_2 = -\sqrt{4 - x^2} \) on the same square screen. (Your circle will appear oval if you do not use a square screen.) See Figure 53.

2.4 Assess Your Understanding

Are You Prepared? Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. To complete the square of \( x^2 + 10x \), you would (add/ subtract) the number _____ (pp. 99–100)
2. Use the Square Root Method to solve the equation \( (x - 2)^2 = 9 \) (pp. 98–99)

Concepts and Vocabulary

3. True or False Every equation of the form \( x^2 + y^2 + ax + by + c = 0 \) has a circle as its graph.
4. For a circle, the _____ is the distance from the center to any point on the circle.
5. True or False The radius of the circle \( x^2 + y^2 = 9 \) is 3.
6. True or False The center of the circle \( (x + 3)^2 + (y - 2)^2 = 13 \) is \((3, -2)\).

Skill Building

In Problems 7–10, find the center and radius of each circle. Write the standard form of the equation.

In Problems 11–20, write the standard form of the equation and the general form of the equation of each circle of radius \( r \) and center \((h, k)\). Graph each circle.

11. \( r = 2; \ (h, k) = (0, 0) \) 12. \( r = 3; \ (h, k) = (0, 0) \) 13. \( r = 2; \ (h, k) = (0, 2) \) 14. \( r = 3; \ (h, k) = (1, 0) \)
15. \( r = 5; \ (h, k) = (4, -3) \) 16. \( r = 4; \ (h, k) = (2, -3) \) 17. \( r = 4; \ (h, k) = (-2, 1) \) 18. \( r = 7; \ (h, k) = (-5, -2) \)
19. \( r = \frac{1}{2}; \ (h, k) = \left(\frac{1}{2}, 0\right) \) 20. \( r = \frac{1}{2}; \ (h, k) = \left(0, \frac{1}{2}\right) \)

In Problems 21–34, (a) find the center \((h, k)\) and radius \( r \) of each circle; (b) graph each circle; (c) find the intercepts, if any.
27. \( x^2 + y^2 + 4x - 4y - 1 = 0 \)
28. \( x^2 + y^2 - 6x + 2y + 9 = 0 \)
29. \( x^2 + y^2 - x + 2y + 1 = 0 \)
30. \( x^2 + y^2 + x + y - \frac{1}{2} = 0 \)
31. \( 2x^2 + 2y^2 - 12x + 8y - 24 = 0 \)
32. \( 2x^2 + 2y^2 + 8x + 7 = 0 \)
33. \( 2x^2 + 8x + 2y^2 = 0 \)
34. \( 3x^2 + 3y^2 - 12y = 0 \)

In Problems 35–42, find the standard form of the equation of each circle.

35. Center at the origin and containing the point \((-2, 3)\)
36. Center \((1, 0)\) and containing the point \((-3, 2)\)
37. Center \((2, 3)\) and tangent to the \(x\)-axis
38. Center \((-3, 1)\) and tangent to the \(y\)-axis
39. With endpoints of a diameter at \((1, 4)\) and \((-3, 2)\)
40. With endpoints of a diameter at \((4, 3)\) and \((0, 1)\)
41. Center \((-1, 3)\) and tangent to the line \(y = 2\)
42. Center \((4, -2)\) and tangent to the line \(x = 1\)

In Problems 43–46, match each graph with the correct equation.

\[
\begin{align*}
&43. \quad (x - 3)^2 + (y + 3)^2 = 9 \quad (a) \quad (x - 1)^2 + (y - 2)^2 = 4 \quad (b) \quad (x + 1)^2 + (y - 2)^2 = 4 \\
&44. \quad (x - 1)^2 + (y + 2)^2 = 4 \quad (c) \quad (x - 3)^2 + (y - 3)^2 = 9 \quad (d) \quad (x + 3)^2 + (y - 3)^2 = 9
\end{align*}
\]

Applications and Extensions

47. Find the area of the square in the figure.

48. Find the area of the shaded region in the figure assuming the quadrilateral inside the circle is a square.

49. Ferris Wheel  The original Ferris wheel was built in 1893 by Pittsburg, Pennsylvania, bridge builder George W. Ferris. The Ferris wheel was originally built for the 1893 World’s Fair in Chicago, but was also later reconstructed for the 1904 World’s Fair in St. Louis. It had a maximum height of 264 feet and a wheel diameter of 250 feet. Find an equation for the wheel if the center of the wheel is on the \(y\)-axis.

Source: inventors.about.com

50. Ferris Wheel  In 2006, the star of Nanchang (in the Jiangxi province) opened as the world’s largest Ferris wheel. It has a maximum height of 160 meters and a diameter of 153 meters, with one full rotation taking approximately 30 minutes. Find an equation for the wheel if the center of the wheel is on the \(y\)-axis.

Source: AsiaOne Travel

Note: Two other even larger Ferris wheels are reportedly to be completed in Asia by 2008 in time for the 2008 summer Olympics

51. Weather Satellites  Earth is represented on a map of a portion of the solar system so that its surface is the circle with equation \(x^2 + y^2 + 2x + 4y - 4091 = 0\). A weather
satellite circles 0.6 unit above Earth with the center of its

52. The tangent line to a circle may be defined as the line that in-
tersects the circle in a single point, called the point of tan-
gency. See the figure.

53. The Greek Method The Greek method for finding the
equation of the tangent line to a circle uses the fact that at any
point on a circle the lines containing the center and the tan-
gent line are perpendicular (see Problem 52). Use this
method to find an equation of the tangent line to the circle

If the circumference of a circle is what is its radius?

If the equation of the circle is and the equation
of the tangent line is , show that:
(a) \[ r^2(1 + m^2) = b^2 \]

(b) The point of tangency is \[ \left( -\frac{r^2 m}{b}, \frac{r^2}{b} \right) \].

(c) The tangent line is perpendicular to the line containing
the center of the circle and the point of tangency.

54. Use the Greek method described in Problem 53 to
find an equation of the tangent line to the circle

55. Refer to Problem 52. The line is tangent to
a circle at . The line is tangent to the same
circle at . Find the center of the circle.

56. Find an equation of the line containing the centers of the two
circles

and

57. If a circle of radius 2 is made to roll along the x-axis, what is
an equation for the path of the center of the circle?

58. If the circumference of a circle is \(6\pi\), what is its radius?

59. Which of the following equations might have the graph
shown? (More than one answer is possible.)
(a) \((x - 2)^2 + (y + 3)^2 = 13\)
(b) \((x - 2)^2 + (y - 2)^2 = 8\)
(c) \((x - 2)^2 + (y - 3)^2 = 13\)
(d) \((x + 2)^2 + (y - 2)^2 = 8\)
(e) \(x^2 + y^2 - 4x - 9y = 0\)
(f) \(x^2 + y^2 + 4x - 2y = 0\)
(g) \(x^2 + y^2 - 9x - 4y = 0\)
(h) \(x^2 + y^2 - 4x - 4y = 4\)

60. Which of the following equations might have the graph
shown? (More than one answer is possible.)
(a) \((x - 2)^2 + y^2 = 3\)
(b) \((x + 2)^2 + y^2 = 3\)
(c) \(x^2 + (y - 2)^2 = 3\)
(d) \((x + 2)^2 + y^2 = 4\)
(e) \(x^2 + y^2 + 10x + 16 = 0\)
(f) \(x^2 + y^2 + 10x - 2y = 1\)
(g) \(x^2 + y^2 + 9x + 10 = 0\)
(h) \(x^2 + y^2 - 9x - 10 = 0\)

61. Explain how the center and radius of a circle can be used to graph a circle.

Are You Prepared? Answers
1. \(+25\) 2. \(-1, 5\)

2.5 Variation

OBJECTIVES
1. Construct a Model Using Direct Variation (p. 196)
2. Construct a Model Using Inverse Variation (p. 197)
3. Construct a Model Using Joint or Combined Variation (p. 197)
When a mathematical model is developed for a real-world problem, it often involves relationships between quantities that are expressed in terms of proportionality:

- Force is proportional to acceleration.
- When an ideal gas is held at a constant temperature, pressure and volume are inversely proportional.
- The force of attraction between two heavenly bodies is inversely proportional to the square of the distance between them.
- Revenue is directly proportional to sales.

Each of these statements illustrates the idea of variation, or how one quantity varies in relation to another quantity. Quantities may vary directly, inversely, or jointly.

### Construct a Model Using Direct Variation

**Definition**

Let $x$ and $y$ denote two quantities. Then $y$ varies directly with $x$, or $y$ is directly proportional to $x$, if there is a nonzero number $k$ such that

$$y = kx$$

The number $k$ is called the constant of proportionality.

The graph in Figure 54 illustrates the relationship between $y$ and $x$ if $y$ varies directly with $x$ and $k > 0$, $x \geq 0$. Note that the constant of proportionality is, in fact, the slope of the line.

If we know that two quantities vary directly, then knowing the value of each quantity in one instance enables us to write a formula that is true in all cases.

### Mortgage Payments

The monthly payment $p$ on a mortgage varies directly with the amount borrowed $B$. If the monthly payment on a 30-year mortgage is $6.65 for every $1000 borrowed, find a formula that relates the monthly payment $p$ to the amount borrowed $B$ for a mortgage with these terms. Then find the monthly payment $p$ when the amount borrowed $B$ is $120,000.

**Solution**

Because $p$ varies directly with $B$, we know that

$$p = kB$$

for some constant $k$. Because $p = 6.65$ when $B = 1000$, it follows that

$$6.65 = k(1000)$$

Solve for $k$.

Since $p = kB$, we have

$$p = 0.00665B$$

In particular, when $B = 120,000$, we find that

$$p = 0.00665(120,000) = 798$$

Figure 55 illustrates the relationship between the monthly payment $p$ and the amount borrowed $B$.
SECTION 2.5 Variation

Construct a Model Using Inverse Variation

Let \( x \) and \( y \) denote two quantities. Then \( y \) varies inversely with \( x \), or \( y \) is inversely proportional to \( x \), if there is a nonzero constant \( k \) such that

\[
y = \frac{k}{x}
\]

The graph in Figure 56 illustrates the relationship between \( y \) and \( x \) if \( y \) varies inversely with \( x \) and \( k > 0, x > 0 \).

Example 2

Maximum Weight That Can Be Supported by a Piece of Pine

See Figure 57. The maximum weight \( W \) that can be safely supported by a 2-inch by 4-inch piece of pine varies inversely with its length \( l \). Experiments indicate that the maximum weight that a 10-foot-long 2-by-4 piece of pine can support is 500 pounds. Write a general formula relating the maximum weight \( W \) (in pounds) to length \( l \) (in feet). Find the maximum weight \( W \) that can be safely supported by a length of 25 feet.

Solution

Because \( W \) varies inversely with \( l \), we know that

\[
W = \frac{k}{l}
\]

for some constant \( k \). Because \( W = 500 \) when \( l = 10 \), we have

\[
500 = \frac{k}{10}
\]

\[
k = 5000
\]

Since \( W = \frac{k}{l} \), we have

\[
W = \frac{5000}{l}
\]

In particular, the maximum weight \( W \) that can be safely supported by a piece of pine 25 feet in length is

\[
W = \frac{5000}{25} = 200 \text{ pounds}
\]

Figure 58 illustrates the relationship between the weight \( W \) and the length \( l \).

Now Work Problem 31

Construct a Model Using Joint Variation or Combined Variation

When a variable quantity \( Q \) is proportional to the product of two or more other variables, we say that \( Q \) varies jointly with these quantities. Finally, combinations of direct and/or inverse variation may occur. This is usually referred to as combined variation.

Let’s look at an example.
Loss of Heat Through a Wall

The loss of heat through a wall varies jointly with the area of the wall and the difference between the inside and outside temperatures and varies inversely with the thickness of the wall. Write an equation that relates these quantities.

Solution

We begin by assigning symbols to represent the quantities:

\[
L = \text{Heat loss} \quad T = \text{Temperature difference} \\
A = \text{Area of wall} \quad d = \text{Thickness of wall}
\]

Then

\[
L = k \frac{AT}{d}
\]

where \( k \) is the constant of proportionality.

In direct or inverse variation, the quantities that vary may be raised to powers. For example, in the early seventeenth century, Johannes Kepler (1571–1630) discovered that the square of the period of revolution \( T \) around the Sun varies directly with the cube of its mean distance \( a \) from the Sun. That is, \( T^2 = ka^3 \), where \( k \) is the constant of proportionality.

Force of the Wind on a Window

The force \( F \) of the wind on a flat surface positioned at a right angle to the direction of the wind varies jointly with the area \( A \) of the surface and the square of the speed \( v \) of the wind. A wind of 30 miles per hour blowing on a window measuring 4 feet by 5 feet has a force of 150 pounds. See Figure 59. What is the force on a window measuring 3 feet by 4 feet caused by a wind of 50 miles per hour?

Solution

Since \( F \) varies jointly with \( A \) and \( v^2 \), we have

\[
F = kAv^2
\]

where \( k \) is the constant of proportionality. We are told that \( F = 150 \) when \( A = 4 \times 5 = 20 \) and \( v = 30 \). Then we have

\[
150 = k(20)(900) \quad F = kAv^2, \ A = 20, \ v = 30
\]

\[
k = \frac{1}{120}
\]

Since \( F = kAv^2 \); we have

\[
F = \frac{1}{120}Av^2
\]

For a wind of 50 miles per hour blowing on a window whose area is \( A = 3 \times 4 = 12 \) square feet, the force \( F \) is

\[
F = \frac{1}{120}(12)(2500) = 250 \text{ pounds}
\]
2.5 Assess Your Understanding

Concepts and Vocabulary

1. If $x$ and $y$ are two quantities, then $y$ is directly proportional to $x$ if there is a nonzero number $k$ such that $y = kx$.

2. True or False  If $y$ varies directly with $x$, then $y = \frac{k}{x}$, where $k$ is a constant.

Skill Building

In Problems 3–14, write a general formula to describe each variation.

3. $y$ varies directly with $x$; $y = 2$ when $x = 10$

4. $v$ varies directly with $t$; $v = 16$ when $t = 2$

5. $A$ varies directly with $x^2$; $A = 4\pi$ when $x = 2$

6. $V$ varies directly with $x^3$; $V = 36\pi$ when $x = 3$

7. $F$ varies inversely with $d^2$; $F = 10$ when $d = 5$

8. $y$ varies inversely with $\sqrt{x}$; $y = 4$ when $x = 9$

9. $z$ varies directly with the sum of the squares of $x$ and $y$; $z = 5$ when $x = 3$ and $y = 4$

10. $T$ varies jointly with the cube root of $x$ and the square of $d$; $T = 18$ when $x = 8$ and $d = 3$

11. $M$ varies directly with the square of $d$ and inversely with the square root of $x$; $M = 24$ when $x = 9$ and $d = 4$

12. $z$ varies directly with the sum of the cubes of $x$ and $y$; $z = 1$ when $x = 2$ and $y = 3$

13. The square of $T$ varies directly with the cube of $a$ and inversely with the square of $d$; $T = 2$ when $a = 2$ and $d = 4$

14. The cube of $z$ varies directly with the sum of the squares of $x$ and $y$; $z = 2$ when $x = 9$ and $y = 4$

Applications and Extensions

In Problems 15–20, write an equation that relates the quantities.

15. Geometry  The volume $V$ of a sphere varies directly with the cube of its radius $r$. The constant of proportionality is $\frac{4\pi}{3}$.

16. Geometry  The square of the length of the hypotenuse $c$ of a right triangle varies jointly with the sum of the squares of the lengths of its legs $a$ and $b$. The constant of proportionality is 1.

17. Geometry  The area $A$ of a triangle varies jointly with the lengths of the base $b$ and the height $h$. The constant of proportionality is $\frac{1}{2}$.

18. Geometry  The perimeter $p$ of a rectangle varies jointly with the sum of the lengths of its sides $l$ and $w$. The constant of proportionality is 2.

19. Physics: Newton’s Law  The force $F$ (in newtons) of attraction between two bodies varies jointly with their masses $m$ and $M$ (in kilograms) and inversely with the square of the distance $d$ (in meters) between them. The constant of proportionality is $G = 6.67 \times 10^{-11}$.

20. Physics: Simple Pendulum  The period of a pendulum is the time required for one oscillation; the pendulum is usually referred to as simple when the angle made to the vertical is less than $5\degree$. The period $T$ of a simple pendulum (in seconds) varies directly with the square root of its length $l$ (in feet).

The constant of proportionality is $\frac{2\pi}{\sqrt{32}}$.

21. Mortgage Payments  The monthly payment $p$ on a mortgage varies directly with the amount borrowed $B$. If the monthly payment on a 30-year mortgage is $6.49 for every $1000 borrowed, find a linear equation that relates the monthly payment $p$ to the amount borrowed $B$ for a mortgage with the same terms. Then find the monthly payment $p$ when the amount borrowed $B$ is $145,000.

22. Mortgage Payments  The monthly payment $p$ on a mortgage varies directly with the amount borrowed $B$. If the monthly payment on a 15-year mortgage is $8.99 for every $1000 borrowed, find a linear equation that relates the monthly payment $p$ to the amount borrowed $B$ for a mortgage with the same terms. Then find the monthly payment $p$ when the amount borrowed $B$ is $175,000.

23. Physics: Falling Objects  The distance $s$ that an object falls is directly proportional to the square of the time $t$ of the fall. If an object falls 16 feet in 1 second, how far will it fall in 3 seconds? How long will it take an object to fall 64 feet?

24. Physics: Falling Objects  The velocity $v$ of a falling object is directly proportional to the time $t$ of the fall. If, after 2 seconds, the velocity of the object is 64 feet per second, what will its velocity be after 3 seconds?

25. Physics: Stretching a Spring  The elongation $E$ of a spring balance varies directly with the applied weight $W$ (see the figure). If $E = 3$ when $W = 20$, find $E$ when $W = 15$. 

\[ W \] 

\[ E \]
26. **Physics: Vibrating String**  The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 48 inches long and vibrates 256 times per second, what is the length of a string that vibrates 576 times per second?

27. **Revenue Equation**  At the corner Shell station, the revenue \( R \) varies directly with the number \( g \) of gallons of gasoline sold. If the revenue is $47.40 when the number of gallons sold is 12, find a linear equation that relates revenue \( R \) to the number \( g \) of gallons of gasoline. Then find the revenue \( R \) when the number of gallons of gasoline sold is 10.5.

28. **Cost Equation**  The cost \( C \) of chocolate-covered almonds varies directly with the number \( A \) of pounds of almonds purchased. If the cost is $23.75 when the number of pounds of chocolate-covered almonds purchased is 5, find a linear equation that relates the cost \( C \) to the number \( A \) of pounds of almonds purchased. Then find the cost \( C \) when the number of pounds of almonds purchased is 3.5.

29. **Demand**  Suppose that the demand \( D \) for candy at the movie theater is inversely related to the price \( p \).
   (a) When the price of candy is $2.75 per bag, the theater sells 156 bags of candy. Express the demand for candy in terms of its price.
   (b) Determine the number of bags of candy that will be sold if the price is raised to $3 a bag.

30. **Driving to School**  The time \( t \) that it takes to get to school varies inversely with your average speed \( s \).
   (a) Suppose that it takes you 40 minutes to get to school when your average speed is 30 miles per hour. Express the driving time to school in terms of average speed.
   (b) Suppose that your average speed to school is 40 miles per hour. How long will it take you to get to school?

31. **Pressure**  The volume of a gas \( V \) held at a constant temperature in a closed container varies inversely with its pressure \( P \). If the volume of a gas is 600 cubic centimeters (cm\(^3\)) when the pressure is 150 millimeters of mercury (mm Hg), find the volume when the pressure is 200 mm Hg.

32. **Resistance**  The current \( i \) in a circuit is inversely proportional to its resistance \( Z \) measured in ohms. Suppose that when the current in a circuit is 30 amperes the resistance is 8 ohms. Find the current in the same circuit when the resistance is 10 ohms.

33. **Weight**  The weight of an object above the surface of Earth varies inversely with the square of the distance from the center of Earth. If Maria weighs 125 pounds when she is on the surface of Earth (3960 miles from the center), determine Maria’s weight if she is at the top of Mount McKinley (3.8 miles from the surface of Earth).

34. **Intensity of Light**  The intensity \( I \) of light (measured in foot-candles) varies inversely with the square of the distance from the bulb. Suppose that the intensity of a 100-watt light bulb at a distance of 2 meters is 0.075 foot-candle. Determine the intensity of the bulb at a distance of 5 meters.

35. **Geometry**  The volume \( V \) of a right circular cylinder varies jointly with the square of its radius \( r \) and its height \( h \). The constant of proportionality is \( \pi \). See the figure. Write an equation for \( V \).

36. **Geometry**  The volume \( V \) of a right circular cone varies jointly with the square of its radius \( r \) and its height \( h \). The constant of proportionality is \( \frac{\pi}{3} \). See the figure. Write an equation for \( V \).

37. **Weight of a Body**  The weight of a body above the surface of Earth varies inversely with the square of the distance from the center of Earth. If a certain body weighs 55 pounds when it is 3960 miles from the center of Earth, how much will it weigh when it is 3965 miles from the center?

38. **Force of the Wind on a Window**  The force exerted by the wind on a plane surface varies jointly with the area of the surface and the square of the velocity of the wind. If the force on an area of 20 square feet is 11 pounds when the wind velocity is 22 miles per hour, find the force on a surface area of 47.125 square feet when the wind velocity is 36.5 miles per hour.

39. **Horsepower**  The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute, rpm) and the cube of its diameter. If a shaft of a certain material 2 inches in diameter can transmit 36 hp at 75 rpm, what diameter must the shaft have in order to transmit 45 hp at 125 rpm?

40. **Chemistry: Gas Laws**  The volume \( V \) of an ideal gas varies directly with the temperature \( T \) and inversely with the pressure \( P \). Write an equation relating \( V, T, \) and \( P \) using \( k \) as the constant of proportionality. If a cylinder contains oxygen at a temperature of 300 K and a pressure of 15 atmospheres in a volume of 100 liters, what is the constant of proportionality \( k \)? If a piston is lowered into the cylinder, decreasing the volume occupied by the gas to 80 liters and raising the temperature to 310 K, what is the gas pressure?

41. **Physics: Kinetic Energy**  The kinetic energy \( K \) of a moving object varies jointly with its mass \( m \) and the square of its velocity \( v \). If an object weighing 25 kilograms and moving with a velocity of 10 meters per second has a kinetic energy of 1250 joules, find its kinetic energy when the velocity is 15 meters per second.
42. **Electrical Resistance of a Wire** The electrical resistance of a wire varies directly with the length of the wire and inversely with the square of the diameter of the wire. If a wire 432 feet long and 4 millimeters in diameter has a resistance of 1.24 ohms, find the length of a wire of the same material whose resistance is 1.44 ohms and whose diameter is 3 millimeters.

43. **Measuring the Stress of Materials** The stress in the material of a pipe subject to internal pressure varies jointly with the internal pressure and the internal diameter of the pipe and inversely with the thickness of the pipe. The stress is 100 pounds per square inch when the diameter is 5 inches, the thickness is 0.75 inch, and the internal pressure is 25 pounds per square inch. Find the stress when the internal pressure is 40 pounds per square inch if the diameter is 8 inches and the thickness is 0.50 inch.

44. **Safe Load for a Beam** The maximum safe load for a horizontal rectangular beam varies jointly with the width of the beam and the square of the thickness of the beam and inversely with its length. If an 8-foot beam will support up to 750 pounds when the beam is 4 inches wide and 2 inches thick, what is the maximum safe load in a similar beam 10 feet long, 6 inches wide, and 2 inches thick?

### Discussion and Writing

45. In the early 17th century, Johannes Kepler discovered that the square of the period $T$ varies directly with the cube of its mean distance $a$ from the Sun. Go to the library and research this law and Kepler’s other two laws. Write a brief paper about these laws and Kepler’s place in history.

46. Using a situation that has not been discussed in the text, write a real-world problem that you think involves two variables that vary directly. Exchange your problem with another student’s to solve and critique.

### CHAPTER REVIEW

#### Things to Know

**Formulas**

- Distance formula (p. 157)
- Midpoint formula (p. 160)
- Slope (p. 174)
- Parallel lines (p. 182)
- Perpendicular lines (p. 183)
- Direct variation (p. 196)
- Inverse variation (p. 197)

**Equations of Lines and Circles**

- Vertical line (p. 177)
- Horizontal line (p. 179)
- Point-slope form of the equation of a line (p. 178)
- Slope-intercept form of the equation of a line (p. 179)
- General form of the equation of a line (p. 181)
- Standard form of the equation of a circle (p. 190)
- Equation of the unit circle (p. 190)
- General form of the equation of a circle (p. 192)

#### Objectives

<table>
<thead>
<tr>
<th>Section</th>
<th>You should be able to ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>1 Use the distance formula (p. 157)</td>
</tr>
<tr>
<td></td>
<td>2 Use the midpoint formula (p. 159)</td>
</tr>
</tbody>
</table>

#### Review Exercises

1(a)–6(a), 48, 49(a), 50
1(b)–6(b), 50

(continued)
2.2 1 Graph equations by plotting points (p. 163)  
2 Find intercepts from a graph (p. 165)  
3 Find intercepts from an equation (p. 166)  
4 Test an equation for symmetry with respect to the x-axis, the y-axis, and the origin (p. 167)  
5 Know how to graph key equations (p. 169)

2.3 1 Calculate and interpret the slope of a line (p. 174)  
2 Graph lines given a point and the slope (p. 176)  
3 Find the equation of a vertical line (p. 177)  
4 Use the point–slope form of a line; identify horizontal lines (p. 178)  
5 Find the equation of a line given two points (p. 179)  
6 Write the equation of a line in slope–intercept form (p. 179)  
7 Identify the slope and y-intercept of a line from its equation (p. 180)  
8 Graph lines written in general form using intercepts (p. 181)

2.4 1 Write the standard form of the equation of a circle (p. 189)  
2 Graph a circle (p. 191)  
3 Work with the general form of the equation of a circle (p. 192)

2.5 1 Construct a model using direct variation (p. 196)  
2 Construct a model using inverse variation (p. 197)  
3 Construct a model using joint or combined variation (p. 197)

Review Exercises

In Problems 1–6, find the following for each pair of points:  
(a) The distance between the points  
(b) The midpoint of the line segment connecting the points  
(c) The slope of the line containing the points  
(d) Interpret the slope found in part (c)

1. (0, 0); (4, 2)  
2. (0, 0); (−4, 6)  
3. (1, −1); (−2, 3)  
4. (−2, 2); (1, 4)  
5. (4, −4); (4, 8)  
6. (−3, 4); (2, 4)

7. Graph \( y = x^2 + 4 \) by plotting points.

8. List the intercepts of the given graph.

In Problems 9–16, list the intercepts and test for symmetry with respect to the x-axis, the y-axis, and the origin.

9. \( 2x = 3y^2 \)  
10. \( y = 5x \)  
11. \( x^2 + 4y^2 = 16 \)  
12. \( 9x^2 - y^2 = 9 \)  
13. \( y = x^4 + 2x^2 + 1 \)  
14. \( y = x^3 - x \)  
15. \( x^2 + x + y^2 + 2y = 0 \)  
16. \( x^2 + 4x + y^2 - 2y = 0 \)

In Problems 17–20, find the standard form of the equation of the circle whose center and radius are given.

17. \((h, k) = (−2, 3); r = 4 \)  
18. \((h, k) = (3, 4); r = 4 \)  
19. \((h, k) = (−1, −2); r = 1 \)  
20. \((h, k) = (2, −4); r = 3 \)

In Problems 21–26, find the center and radius of each circle. Graph each circle. Find the intercepts, if any, of each circle.

21. \( x^2 + (y - 1)^2 = 4 \)  
22. \( (x + 2)^2 + y^2 = 9 \)  
23. \( x^2 + y^2 - 2x + 4y - 4 = 0 \)  
24. \( x^2 + y^2 + 4x - 4y - 1 = 0 \)  
25. \( 3x^2 + 3y^2 - 6x + 12y = 0 \)  
26. \( 2x^2 + 2y^2 - 4x = 0 \)
In Problems 27–36, find an equation of the line having the given characteristics. Express your answer using either the general form or the slope–intercept form of the equation of a line, whichever you prefer.

27. Slope = -2; containing the point (3, -1)  
28. Slope = 0; containing the point (-5, 4)  
29. Vertical; containing the point (-3, 4)  
30. x-intercept = 2; containing the point (4, -5)  
31. y-intercept = -2; containing the point (5, -3)  
32. Containing the points (3, -4) and (2, 1)  
33. Parallel to the line 2x - 3y = -4; containing the point (-5, 3)  
34. Parallel to the line x + y = 2; containing the point (1, -3)  
35. Perpendicular to the line x + y = 2; containing the point (4, -3)  
36. Perpendicular to the line 3x - y = -4; containing the point (-2, 4)

In Problems 37–40, find the slope and y-intercept of each line. Graph the line, labeling any intercepts.

37. 4x - 5y = -20  
38. 3x + 4y = 12  
39. \( \frac{1}{2}x - \frac{1}{3}y = \frac{1}{6} \)  
40. \( \frac{3}{4}x + \frac{1}{2}y = 0 \)

In Problems 41–44, find the intercepts and graph each line.

41. 2x - 3y = 12  
42. x - 2y = 8  
43. \( \frac{1}{2}x + \frac{1}{3}y = 2 \)  
44. \( \frac{1}{3}x - \frac{1}{4}y = 1 \)

45. Sketch a graph of \( y = x^2 \).

46. Sketch a graph of \( y = \sqrt{x} \).

47. Graph the line with slope \( \frac{2}{3} \) containing the point (1, 2).

48. Show that the points \( A = (3, 4) \), \( B = (1, 1) \), and \( C = (-2, 3) \) are the vertices of an isosceles triangle.

49. Show that the points \( A = (-2, 0) \), \( B = (-4, 4) \), and \( C = (8, 5) \) are the vertices of a right triangle in two ways:  
(a) By using the converse of the Pythagorean Theorem  
(b) By using the slopes of the lines joining the vertices

50. The endpoints of the diameter of a circle are \( (-3, 2) \) and \( (5, -6) \). Find the center and radius of the circle. Write the standard equation of this circle.

51. Show that the points \( A = (2, 5) \), \( B = (6, 1) \), and \( C = (8, -1) \) lie on a line by using slopes.

52. **Mortgage Payments**  
The monthly payment \( p \) on a mortgage varies directly with the amount borrowed \( R \). If the monthly payment on a 30-year mortgage is $854.00 when $130,000 is borrowed, find an equation that relates the monthly payment \( p \) to the amount borrowed \( B \) for a mortgage with the same terms. Then find the monthly payment \( p \) when the amount borrowed \( B \) is $165,000.

53. **Revenue Function**  
At the corner Esso station, the revenue \( R \) varies directly with the number \( g \) of gallons of gasoline sold. If the revenue is $46.67 when the number of gallons sold is 13, find an equation that relates revenue \( R \) to the number \( g \) of gallons of gasoline. Then find the revenue \( R \) when the number of gallons of gasoline sold is 11.2.

54. **Weight of a Body**  
The weight of a body varies inversely with the square of its distance from the center of Earth. Assuming that the radius of Earth is 3960 miles, how much would a man weigh at an altitude of 1 mile above Earth’s surface if he weighs 200 pounds on Earth’s surface?

55. **Kepler’s Third Law of Planetary Motion**  
Kepler’s Third Law of Planetary Motion states that the square of the period of revolution \( T \) of a planet varies directly with the cube of its mean distance \( a \) from the Sun. If the mean distance of Earth from the Sun is 93 million miles, what is the mean distance of the planet Mercury from the Sun, given that Mercury has a “year” of 88 days?
204  CHAPTER 2  Graphs

56. Create four problems that you might be asked to do given the two points \((-3, 4)\) and \((6, 1)\). Each problem should involve a different concept. Be sure that your directions are clearly stated.

57. Describe each of the following graphs in the \(xy\)-plane. Give justification.
(a) \(x = 0\)    (b) \(y = 0\)    (c) \(x + y = 0\)    (d) \(xy = 0\)    (e) \(x^2 + y^2 = 0\)

58. Suppose that you have a rectangular field that requires watering. Your watering system consists of an arm of variable length that rotates so that the watering pattern is a circle. Decide where to position the arm and what length it should be so that the entire field is watered most efficiently. When does it become desirable to use more than one arm?
[Hint: Use a rectangular coordinate system positioned as shown in the figures. Write equations for the circle(s) swept out by the watering arm(s).]

\[ \text{Square field} \]
\[ \text{Rectangular field, one arm} \]
\[ \text{Rectangular field, two arms} \]

CHAPTER TEST

In Problems 1–3, use \(P_1 = (-1, 3)\) and \(P_2 = (5, -1)\).
1. Find the distance from \(P_1\) to \(P_2\).
2. Find the midpoint of the line segment joining \(P_1\) and \(P_2\).
3. (a) Find the slope of the line containing \(P_1\) and \(P_2\).
   (b) Interpret this slope.
4. Graph \(y = x^2 - 9\) by plotting points.
5. Sketch the graph of \(y^2 = x\).
6. List the intercepts and test for symmetry: \(x^2 + y = 9\).
7. Write the slope–intercept form of the line with slope \(-2\) containing the point \((3, -4)\). Graph the line.
8. Write the general form of the circle with center \((4, -3)\) and radius 5.

9. Find the center and radius of the circle \(x^2 + y^2 + 4x - 2y - 4 = 0\). Graph this circle.
10. For the line \(2x + 3y = 6\), find a line parallel to it containing the point \((1, -1)\). Also find a line perpendicular to it containing the point \((0, 3)\).
11. Resistance due to a Conductor  The resistance (in ohms) of a circular conductor varies directly with the length of the conductor and inversely with the square of the radius of the conductor. If 50 feet of wire with a radius of \(6 \times 10^{-3}\) inch has a resistance of 10 ohms, what would be the resistance of 100 feet of the same wire if the radius is increased to \(7 \times 10^{-3}\) inch?

CUMULATIVE REVIEW

In Problems 1–8, find the real solution(s) of each equation.
1. \(3x - 5 = 0\)
2. \(x^2 - x - 12 = 0\)
3. \(2x^2 - 5x - 3 = 0\)
4. \(x^2 - 2x - 2 = 0\)
5. \(x^2 + 2x + 5 = 0\)
6. \(\sqrt{2x + 1} = 3\)
7. \(|x - 2| = 1\)
8. \(\sqrt{x^2 + 4x} = 2\)

In Problems 9 and 10, solve each equation in the complex number system.
9. \(x^2 = -9\)
10. \(x^2 - 2x + 5 = 0\)

In Problems 11–14, solve each inequality. Graph the solution set.
11. \(2x - 3 \leq 7\)
12. \(-1 < x + 4 < 5\)
13. \(|x - 2| \leq 1\)
14. \(|2 + x| > 3\)
15. Find the distance between the points \(P = (-1, 3)\) and \(Q = (4, -2)\). Find the midpoint of the line segment from \(P\) to \(Q\).

16. Which of the following points are on the graph of \(y = x^3 - 3x + 1\)?
   (a) \((-2, -1)\)    (b) \((2, 3)\)    (c) \((3, 1)\)
17. Sketch the graph of \(y = x^3\).
18. Find the equation of the line containing the points \((-1, 4)\) and \((2, -2)\). Express your answer in slope–intercept form.
19. Find the equation of the line perpendicular to the line \(y = 2x + 1\) and containing the point \((3, 5)\). Express your answer in slope–intercept form and graph the line.
20. Graph the equation \(x^2 + y^2 - 4x + 8y - 5 = 0\).
CHAPTER PROJECT

Predicting Olympic Performance  Measurements of human performance over time sometimes follow a strong linear relationship for reasonably short periods. In 2004 the Summer Olympic Games returned to Greece, the home of both the ancient Olympics and the first modern Olympics. The following data represent the winning times (in hours) for men and women in the Olympic marathon.

<table>
<thead>
<tr>
<th>Year</th>
<th>Men</th>
<th>Women</th>
</tr>
</thead>
<tbody>
<tr>
<td>1984</td>
<td>2.16</td>
<td>2.41</td>
</tr>
<tr>
<td>1988</td>
<td>2.18</td>
<td>2.43</td>
</tr>
<tr>
<td>1992</td>
<td>2.22</td>
<td>2.54</td>
</tr>
<tr>
<td>1996</td>
<td>2.21</td>
<td>2.43</td>
</tr>
<tr>
<td>2000</td>
<td>2.17</td>
<td>2.39</td>
</tr>
</tbody>
</table>

Source: www.hickoksports.com/history/olmtandf.shtml

1. Treating year as the independent variable and the winning value as the dependent variable, find linear equations relating these variables (separately for men and women) using the data for the years 1992 and 1996. Compare the equations and comment on any similarities or differences.

2. Interpret the slopes in your equations from part 1. Do the y-intercepts have a reasonable interpretation? Why or why not?

3. Use your lines to predict the winning time in the 2004 Olympics. Compare your predictions to the actual results (2.18 hours for men and 2.44 hours for women). How well did your equations do in predicting the winning times?

4. Repeat parts 1 to 3 using the data for the years 1996 and 2000. How do your results compare?

5. Would your equations be useful in predicting the winning marathon times in the 2104 Summer Olympics? Why or why not?

6. Pick your favorite Winter Olympics event and find the winning value (that is distance, time, or the like) in two Winter Olympics prior to 2006. Repeat parts 1 to 3 using your selected event and years and compare to the actual results of the 2006 Winter Olympics in Torino, Italy.