The Fibonacci Sequence

The Fibonacci sequence is defined by $a_1 = a_2 = 1$, and by the following recursive relationship: $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. The sequence begins 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... Each term after the second one is obtained by adding the previous two terms.

After 500 years of attempts by others, Leonardo Fibonacci discovered an explicit formula for this sequence around 1200 A.D. :

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

It's not too hard to check this for n = 1, 2, and 3.

The condition $a_n = a_{n-1} + a_{n-2}$ can be divided by a_{n-1} to obtain

$$\frac{a_n}{a_{n-1}} = 1 + \frac{a_{n-2}}{a_{n-1}} \quad .$$

If the limit L= $\lim\limits_{n\to\infty}\frac{a_n}{a_{n-1}}$ exists, then $\lim\limits_{n\to\infty}\frac{a_{n-2}}{a_{n-1}} = \frac{1}{L}$, since $\frac{a_{n-2}}{a_{n-1}}$ is the reciprocal of $\frac{a_{n-1}}{a_{n-2}}$

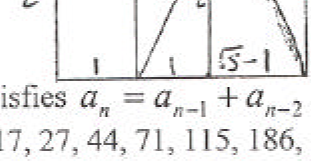
and $\lim\limits_{n\to\infty}\frac{a_{n-1}}{a_{n-2}} = L$. We get $\lim\limits_{n\to\infty}\frac{a_n}{a_{n-1}} = 1 + \lim\limits_{n\to\infty}\frac{a_{n-2}}{a_{n-1}}$.

$$\text{So } L = 1 + \frac{1}{L}. \text{ Or } L^2 = L + 1 \to L^2 - L - 1 = 0 .$$

Solving by the quadratic formula gives:

$L = \frac{1 \pm \sqrt{5}}{2}$. Reject $\frac{1-\sqrt{5}}{2}$, since the limit of a positive sequence can not be negative. So $L = \frac{1+\sqrt{5}}{2}$. (There's that number again!)

Taking the ratio of 233 to 144, say, gives approximately 1.61805556, while L is approximately 1.618033989 . The ratio of any 2 consecutive terms (starting with 3 and 5) is about 1.6 .
The number L is called the Golden Ratio, and it occurs in art and architecture as pleasing to the eye. It is the ratio of the sides of the Golden Rectangle.
Divide a 2 by 2 square into 2 equal parts. Draw a circular sector as shown.



A Fibonacci-type sequence satisfies $a_n = a_{n-1} + a_{n-2}$. Suppose $a_1 = 3$ and $a_2 = 7$.
We get the sequence 3, 7, 10, 17, 27, 44, 71, 115, 186, ... As above, the ratio approaches L.

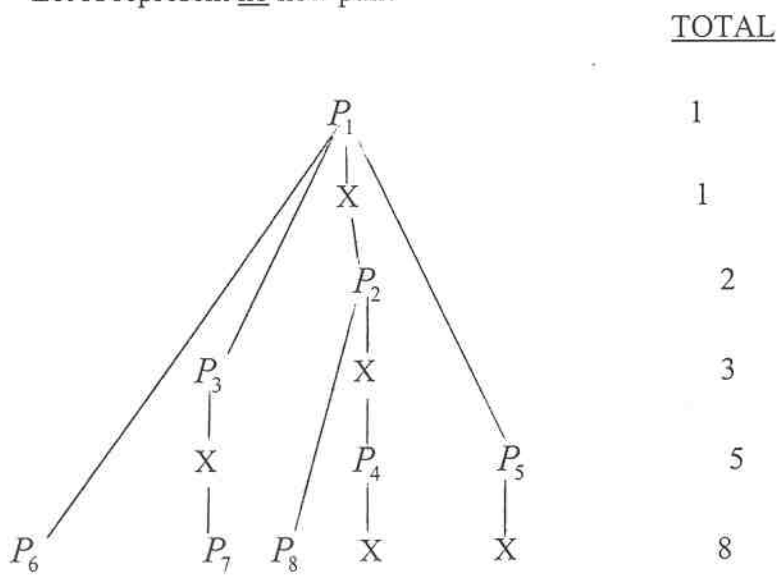
For instance, $\frac{186}{115}$ is approximately 1.617391304 .

A daisy's head has 21 spirals in a clockwise direction, and 34 counterclockwise. These are adjacent Fibonacci numbers. Similarly, pine-cone scales have 5 spirals one way, and 8 the other.

An adult male-female pair of rabbits reproduces another male-female pair after 2 months and a pair every month after that.

Let P_1 = pair one, P_2 = pair two, and so on.

Let X represent <u>no</u> new pair.



Exercise: Convince yourself that after another month, the total will be up to 13. We're getting the Fibonacci sequence.

There is a lot more information about the Fibonacci sequence if you care to research the subject.